A geometric approach to BRST quantization of the bosonic string

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Abstract

BRST quantization find its roots in the years 1974-1976 in papers by Becchi, Rouet, and Stora [1], and independently, by Tyutin [4]. At first, it was a very heuristic method of quantizing gauge theories, but since then, a great work has been done to understand geometrically this procedure. This is what we want to describe in this paper. The first section constructs the BRST cohomology in a simplified framework, and in the second, links are proposed between the simplified mathematical approach and the physical approach in the bosonic string theory. In particular, the BRST symmetry is closely related to the Fadeev-Popov gauge fixing.

1 Geometric BRST quantization

1.1 Symplectic reduction

Let (X, O) be a symplectic manifold on which a connected Lie group G acts by symplectomorphisms, such that there is a moment map:

$$\phi : X \to \mathfrak{g}^*$$

that is, a function satisfying that $\forall x \in X$ and $\forall \xi \in \mathfrak{g}$:

$$d(\phi(x)(\xi)) = \iota_{\tilde{\xi}_x} \Omega$$

where $\tilde{\xi}$ is the fundamental vector field generated by ξ. We can define a map $\delta : \mathfrak{g} \to Fun(X)$ by setting:

$$\delta(\xi)(x) = \phi(x)(\xi)$$

for all $\xi \in \mathfrak{g}$ and $x \in X$. Provided $0 \in \mathfrak{g}^*$ is a regular value of the moment map, we can define:

$$C = \phi^{-1}(0)$$

which is then a closed embedded coisotropic submanifold. Moreover, $\forall x \in X$, $T_x M_0^\perp$ is precisely the subspace of $T_x M_0$ spanned by the Killing vector fields induced by the action of G.

Now define $B = C/G$. B is the *Marsden-Weinstein symplectic reduction* of X. Note that there are two steps in this procedure : one step of restriction from X to C, and one step where one quotients by the group's action. For more details on symplectic reduction and the rest of this section, see [2], [5].

1.2 Lie algebra cohomology

Any smooth function on B induces a function on C which is constant on the fibers, and conversely, any function on C which is constant on the fibers induces a function on B.

Let's highlight that the hamiltonian Killing vector fields Ξ_i generated by the Lie algebra $\mathfrak{g}$ of G form a global basis of the tangent spaces to the fibers. Let's chose a dual basis ω^i of the cotangent space to the fibers, that is, $\omega^i(\Xi_j) = \delta^i_j$, and define:

$$d_V : \mathcal{C}^\infty(C) \to \Omega^1_V(C)$$

by:

$$d_V(f) = \sum_i (\mathcal{L}_{\Xi_i} f) \omega^i$$

where $\mathcal{L}_{\Xi_i} f$ denotes the Lie derivative of f in the direction Ξ_i. One extends this function to a derivation:

$$d_V : \Omega^p_V(C) \to \Omega^{p+1}_V C$$

by defining:

$$d_V \omega^i = -\frac{1}{2} \sum_{j,k} f^i_{jk} \omega^j \wedge \omega^k$$

This function satisfies the relation $d_V \circ d_V = 0$ (it is proved by the following proposition) and thus $(\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(C), d_V)$ form a cochain complex from which one can compute the Lie algebra cohomology.

Proposition 1.1. *The differential d_V coincide with the Chevalley-Eilenberg's differential corresponding to the trivial representation of $\mathfrak{g}$ on $\mathbb{R}$.*

Proof. Let x_a and x_b be in $\mathfrak{g}$. One computes:

$$d_V \omega^i(x_a, x_b) = -\frac{1}{2} \sum_{j,k} f^i_{jk}(\delta^j_a \delta^k_b - \delta^j_b \delta^k_a)$$

$$d_V \omega^i(x_a, x_b) = -f^i_{ab}$$

$$d_V \omega^i(x_a, x_b) = (-1)^{1+2} \omega^i([x_a, x_b])$$

$\square$

There are interpretations of the low-degree cohomology groups of the Chevalley-Eilenberg complex. For example, $H^0_V(\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(C))$ is the subspace of $\mathcal{C}^\infty(C)$ constituted by all functions which are constant on the fibers.

1.3 BRST cohomology

We found a cohomology describing the property of being a scalar function under a gauge change, but unfortunately it is defined on $\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(C)$, whereas one want to work on globally defined objects, that is, with $\mathcal{C}^\infty(X)$ instead of $\mathcal{C}^\infty(C)$. C is defined defined by the constraints $C = \phi^{-1}(0)$ which corresponds to the gauge fixing conditions. Despite these constraints, we'd like to work on globally defined objects.

Suppose that:

$$\mathcal{C}^\infty(C) = \mathcal{C}^\infty(X)/I$$

where I is some ideal in $\mathcal{C}^\infty(X)$. By the very definition of C, $\forall \xi \in \mathfrak{g}$, $\delta x \in I$. Suppose that I is generated by the $\delta(\xi)$, that is:

$$I = \mathcal{C}^\infty(X)\delta(\mathfrak{g})$$

Now one can extend δ as a linear anti-derivation on $\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(X)$ by letting $\delta(\xi \otimes 1) = 1 \otimes \delta(\xi)$ and $\delta(1 \otimes f) = 0$. We thus have a complex:

$$\to \bigwedge^k \mathfrak{g} \otimes \mathcal{C}^\infty(X) \to \cdots \to \mathfrak{g} \otimes \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$$

with differential δ. It is called the Koszul complex of $\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(X)$. The Koszul construction aim to algebraically represent the quotient:

$$\mathcal{C}^\infty(C) = \mathcal{C}^\infty(X)/I$$

Its k-th homology space is indeed zero for $k > 0$, and:

$$H^\delta_0(\bigwedge \mathfrak{g} \otimes \mathcal{C}^\infty(X)) = \mathcal{C}^\infty(X)/I$$

Let's add the Lie algebra cohomology to obtain:

$$\bigwedge^{p,q} \mathfrak{g} \otimes \mathcal{C}^\infty(X) \xrightarrow{\delta} \bigwedge^{p,q-1} \mathfrak{g} \otimes \mathcal{C}^\infty(X)$$
$$\downarrow{d_V}$$
$$\bigwedge^{p+1,q} \mathfrak{g} \otimes \mathcal{C}^\infty(X)$$

where we wrote: $\bigwedge^{p,q} \mathfrak{g} = \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g}$.

Since $\delta \circ d = d \circ \delta$, it is a double complex. Let:

$$P_m = \bigoplus_{p-q=m} \bigwedge^p \mathfrak{g}^* \otimes \bigwedge^q \mathfrak{g} \otimes \mathcal{C}^\infty(X)$$

and:

$$D_m : P_m \to P_{m+1}$$

be given by:

$$D_m(.) = d(.) + (-1)^m \delta(.)$$

Eventually we have the following cochain complex:

$$\cdots \xrightarrow{D_{m-1}} P_m \xrightarrow{D_m} P_{m+1} \xrightarrow{D_{m+1}} \cdots$$

which computes the classical BRST cohomology. It can be proved ([2]) that $H^n_D = 0$ if $n < 0$ and $H^n_D = H^n_V(C)$ if $n \geq 0$.

That was exactly the purpose of the BRST procedure : this cohomology allows to work on globally defined objects without loosing the profound sense of gauge.

In physics literature, elements of $\bigwedge \mathfrak{g}^*$ are called *ghosts*, those of $\bigwedge \mathfrak{g}$ are called *antighosts*, while $m = p - q$ is the *ghost number*.

Now let's recall Fadeev-Popov gauge fixing for the bosonic string action, and try to sketch the links between the geometrical approach and the path-integral approach, and in particular the nature and role of ghost fields.

2 BRST string quantization

In the framework of field theory (especially for string theory), the symplectic manifold is a space of fields and is not finite-dimensional anymore. Yet, we won't bother with such considerations and will make direct links, even if it implies some loss of mathematical rigor.

2.1 Fadeev-Popov gauge fixing

Let (M, g) be the two-dimensional world-sheet Riemannian manifold, and $X^\mu(\sigma^1, \sigma^2)$, $\mu = 0, \cdots, (d-1)$, be d embeddings of M into $(\mathbb{R}^d, \eta)$ where $\eta_{\mu\nu}$ is Minkowski's metric in d dimensions. Starting from Polyakov's action:

$$S[X, g] = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} g^{ab} \partial_a X^\mu \partial_b X_\mu$$

one defines the following path-integral:

$$Z = \int \frac{[dX][dg]}{V_{gauge}} \exp(-S[X, g])$$

Here, the gauge group G is the group of reparametrizations of the world-sheet, as well as Weyl's transformations of the metric. G acts on the space of metrics on M by taking the pull-back by any reparametrization of M and combining it with a Weyl transformation of parameter ω. Let $\zeta \in G$, then:

$$(\zeta \cdot g)_{\alpha\beta}(\sigma') = g^\zeta_{ab}(\sigma') = e^{2\omega(\sigma)} \frac{\partial\sigma^c}{\partial\sigma^a} \frac{\partial\sigma^d}{\partial\sigma^b} g_{cd}(\sigma)$$

In the Z path integral, one has to divide by the gauge group's volume, in order to take into account the fact that the space of fields is redundant, in the sense that two different configurations of fields on M can correspond to the same physical system. This is the sense of the gauge group's action on this space of fields: all orbits correspond to the same physical system. We can use the gauge freedom to fix some fields, for example, the metric, to a canonical form. Unfortunately, there may remain some freedom, that is some non trivial gauge transformation which leave the metric invariant, thus one has to take the quotient of the submanifold of the space of fields defined by the gauge-fixing conditions. To put it in a nutshell, in order to have a well-defined path-integral, one has to take a symplectic reduction of the symplectic manifold of all fields on M. Such a space is in a one-to-one correspondance with the space of all physically-distinguishable configurations.

For g some metric on M, one writes:

$$1 = \Delta_{FP}(g) \int [d\zeta] \delta(g - \hat{g}^\zeta)$$

where $\hat{g}$ is the trivial, or fiducial, metric $\hat{g}_{ab} = \delta_{ab}$. The delta function ensures that the only transformations which will be kept are the ζ which bring the fiducial metric to g. This step corresponds to the gauge fixing, or, to stick to the notations of the previous section, to restrict oneself from X to C.

One can insert the former expression into Z:

$$Z[\hat{g}] = \int \frac{[dX dg d\zeta]}{V_{gauge}} \Delta_{FP}(g) \delta(g - \hat{g}^\zeta) e^{-S[X, g]}$$

One may then integrate over g, and shift the integration variable $X \to X^\zeta$ to get:

$$Z[\hat{g}] = \int \frac{[d\zeta dX^\zeta]}{V_{gauge}} \Delta_{FP}(\hat{g}^\zeta) e^{-S[X^\zeta, \hat{g}^\zeta]}$$

By definition of the gauge group:

$$S[X^\zeta, \hat{g}^\zeta] = S[X, \hat{g}]$$

and because:

$$\int [d\zeta'] \delta(g^\zeta - \hat{g}^{\zeta'}) = \int [d\zeta'] \delta(g - \hat{g}^{\zeta^{-1}\zeta'})$$

one also has: $\Delta_{FP}(\hat{g}^\zeta) = \Delta_{FP}(\hat{g})$ for every ζ. Thus, one can integrate over the fields ζ.

This integration is trivial and only gives the volume of the gauge group, because one had "three" gauge functions, namely the reparametrization of the world-sheet by changing the two coordinates on M and the local rescaling of the metric, and the gauge-fixing functions fix the three coordinates of the metric on M. Unrigorously, it shows that the measure of the remaining subgroup of G which leaves the metric invariant is a measure zero subgroup.

Eventually, ons is led to:

$$Z[\hat{g}] = \int [dX] \Delta_{FP}(\hat{g}) \exp(-S[X, \hat{g}])$$

which is a better-defined path-integral, because the gauge has already been fixed. Now, the last thing to do is to obtain the expression of $\Delta_{FP}(\hat{g})$. One could say that this function is an induced measure on the submanifold C of X. Because of the expression of $\Delta_{FP}(\hat{g})^{-1}$, one is going to consider a path-integral on all gauge transformation which leave the metric invariant. It's equivalent to look at all Killing vectors, that is, fundamental vector fields induced by vectors in the Lie algebra $\mathfrak{g}$ of G or infinitesimal transformations, under the flow of whose the form metric does not change.

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One writes:

$$\delta g_{ab} = 2\delta\omega g_{ab} - \nabla_a\delta\sigma_b - \nabla_b\delta\sigma_a$$

we could also write $\Delta_{FP}(\hat{g})^{-1}$ under the form:

$$\int [d\delta\omega d\delta\sigma]\delta(-(2\delta\omega - \hat{\nabla}\cdot\delta\sigma)\hat{g}_{ab} + 2\hat{P}_1\delta\sigma)$$

where:

$$(\hat{P}_1\delta\sigma)_{ab} = \frac{1}{2}(\nabla_a\delta\sigma_b + \nabla_b\delta\sigma_a - \hat{g}_{ab}\nabla_c\delta\sigma^c)$$

After expressing the delta functional as an exponential, and integrating over $\delta\omega$, one is left with:

$$\int [d\beta' d\delta\sigma]exp\{4\pi i \int d^2\sigma\sqrt{(\hat{g})}\beta'^{ab}(\hat{P}_1\delta\sigma)_{ab}\}$$

for the expression of $\Delta_{FP}(\hat{g})^{-1}$, where the β's are traceless symmetric tensors.

This determinant can be inverted be replacing all bosonic fields by a corresponding ghost field: $\delta\sigma^a \to c^a$ and $\beta'_{ab} \to b_{ab}$. One is left with the gauge-fixed path integral :

$$Z = \int [dXdcdbdB]exp\{-S - S_{gh} - S_{G.F}\}$$

Ghost fields c and b have a very natural interpretation in the geometrical framework: on the one hand, c-fields (or ghosts) are related to a global basis of the dual of the Lie algebra $\mathfrak{g}$. If one BRST-quantizes an abstract theory, at the end there should be one c-field for each vector of a basis of $\mathfrak{g}$, that is, one c-field for each infinitesimal transformation of the gauge group G. On the other, b-fields (or anti-ghosts) are related to the gauge-fixing conditions, there is one b-field for each constraint relation.

2.2 Bosonic string BRST symmetry

So one can class the BRST transformations of the bosonic string theory into two distinct subgroups: the first one contains the transformations:

$$\delta_B h_{ab} = -i\epsilon(2c_\omega h_{ab} - \nabla_a c_b - \nabla_b c_a)$$

$$\delta_B X^\mu = -\epsilon(c^a\partial_a X^\mu)$$

$$\delta_B c^a = -i\epsilon c^b\partial_b c^a$$

$$\delta_B c^\omega = -i\epsilon c^b\partial_b c^\omega$$

and is related to the Lie algebra cohomology, and the quotient of the space of fields by the gauge group action, while

$$\delta_B B_{ab} = 0$$

$$\delta_B b_{ab} = \epsilon B_{ab}$$

are related to the Koszul complex, and thus to the restriction to the submanifold defined by the constraints on the form of the metric.

To conclude our geometric insight of BRST symmetry in the bosonic string theory, let's make a two remarks.

First, since $H_B^0 = H_V^0(C)$, the initial action functional S, which is gauge invariant and thus in $H_V^0(C)$, is identified in this cohomology space with all functional which differ only by exact terms. For example, adding the $S_2 + S_3$ term doesn't change the equivalence class one is looking at, since:

$$S_2 + S_3 = i\epsilon^{-1}\delta_B(b_{ab}F^{ab}(g))$$

The second important idea we'd like to put in exergue is the notion of *physical state*. These are special states, those which are meaningful from an experimental point of view. In particular, different choices of gauges should not affect the scattering amplitudes of the theory. In the quantized theory, fields are operators acting on the states of the theory. The BRST symmetry is a transformation of the operators, but since we can express it in terms of an anti-commutator with a BRST operator Q_B, which acts on the states as well. This operator Q_B has the same properties as the BRST differential δ_B, and in particular, is nilpotent of order 2 and induces a cochain complex of Hilbert spaces, where the k-th cochain space is the set of all states of ghost number k. As done in Polchinski's book [3], it can be shown that physical states are closed in the sense that $|\Psi\rangle$ is physical if:

$$Q_B|\psi\rangle = 0$$

On the other hand, states which are exact for Q_B are called *null states*. Null states are orthogonal to every physical state. If one wants to avoid redundancies, one can quotient the closed states by the exact states, and thus the spectrum of the theory is obtained as the direct sum of the cohomology groups of the Q_B cohomology on the states. The spectrum obtained this way is in fact equivalent to

the spectra one has after quantizing the classical theory by means of the *Old Covariant Quantization*, or *Light-cone Quantization* [3], but BRST quantization corresponds to this great philosophical idea of directly relating the BRST cohomology as seen as symplectic reduction of the phase space, and BRST cohomology as seen as the algebraic way of understanding physical states.

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