Master thesis

Description of super-Teichmüller spaces as moduli spaces of graph connections

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Abstract

Let S be a topological (i.e differentiable) surface, and let $\mathrm{Diff}^0(S)$ the connected component of identity in the group of diffeomorphisms of S. The classical Teichmüller space (or $\mathrm{PSL}_2(\mathbb{R})$-Teichmüller space) $\mathcal{T}(S)$ of S is defined as the space of complex structures of S modulo $\mathrm{Diff}^0(S)$.

If S in regular enough in some sense, hyperbolic and with at least one puncture, then any triangulation Γ of S defines special coordinates on $\mathcal{T}(S)$ called *shear coordinates*. These coordinates are the parameters which prescribe the complex gluing of the triangles of Γ modulo $\mathrm{Diff}^0(S)$.

We first define ciliated surfaces, which are nice open surfaces having possibly a finite set of marked point on their boundary. We recall the construction of the Teichmüller $\mathcal{X}$-space $\mathcal{T}^x(S)$ of ciliated surfaces (which is a $2^n : 1$ branched cover of $\mathcal{T}(S)$) and study some of its properties as well as how it is related to $\mathcal{T}(S)$.

The coordinates on $\mathcal{T}^x(S)$ can also be described as the gauge invariants of the $\mathbb{R}^*$-connections on a graph which is constructed thanks to a triangulation of S. We try to give some rigour to those ideas.

Then we recall the construction of algebraic and smooth super-manifolds. We introduce the group $\mathrm{SpO}(2|1)(\mathbb{R})$ which arise in the super-generalisation of Teichmüller spaces.

The last part is devoted to the study of configuration spaces of tuples of points in $\mathbb{P}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$, and the construction of a gauge theory on a graph which is an equivalent description of the super-Teichmüller space of a ciliated surface.

Introduction

Given a hyperbolic ciliated surface S with at least one boundary component, there are *a priori* more than one inequivalent complex structures on S. The Teichmüller spaces $\mathcal{T}(S)$ and $\mathcal{T}^x(S)$ are geometrical classification spaces of those complex structures. These spaces have an intrinsic theoretical interest, but also a lot of interesting properties. For example, they can be endowed with a structure of real Poisson manifold (with of without corners). They are also topologically trivial.

After the choice of a triangulation Γ of S, one can construct special coordinates called *shear coordinates* on them. Shear coordinates are invariants of tuples of points in $\mathbb{RP}^1$ under the action of $\mathrm{PSL}_2(\mathbb{R})$. This action is induced by the standard action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^2$, hence these invariants can be described as a gauge theory in which the orbits of tuples of points in $\mathbb{R}^2$ under the action of $\mathrm{PSL}_2(\mathbb{R})$ correspond bijectively to the $\mathbb{R}^*$-connections on a graph, and with the gauge freedom corresponding the rescaling of every vector in $\mathbb{R}^2$ by some invertible real number. Hence the gauge invariants are projective invariants under the action $\mathrm{PSL}_2(\mathbb{R})$.

The construction of the Teichmüller spaces can be generalised to a super-geometrical framework, giving the super-Teichmüller spaces of a hyperbolic ciliated surface with at least one puncture. One can again define shear coordinates on those generalised Teichmüller spaces. We study a possible construction of a gauge theory giving these coordinates as gauge invariants, as in the classical case.

The first part of this work is a recollection of the construction of shear coordinates on the Teichmüller spaces $\mathcal{T}(S)$ and $\mathcal{T}^x(S)$, and of some of their properties. The second part is devoted to the construction of the gauge theory on surface graphs in which shear coordinates appear as gauge invariants. Meanwhile we will study different definitions of the classical cross-ratio of four points in $\mathbb{RP}^1$ and the links between them. Then we recall the grounds of super-geometry in order to define the space $\mathbb{P}^{1|1}$ which is a super-generalisation of the projective line $\mathbb{RP}^1$, and the group $\mathrm{SpO}(2|1)(\mathbb{R})$ which is a super-generalisation of $\mathrm{PSL}_2(\mathbb{R})$ acting on $\mathbb{P}^{1|1}$. Whereas any two positively oriented triples of points in $\mathbb{RP}^1$ can be mapped one to another by some element of $\mathrm{PSL}_2(\mathbb{R})$, orbits of positively oriented triples of points in $\mathbb{P}^{1|1}$ are parametrised by $G(\mathbb{R})_1/\{\pm 1\}$. We compute the odd invariant of a triple of point in $\mathbb{P}^{1|1}$ explicitly and study its interesting properties. Eventually, we construct an enhanced gauge theory on surface graphs giving the invariants of tuples of points in $\mathbb{P}^{1|1}$ as its gauge invariants.

I am very thankful to Vladimir Fock for the energy and time he spent on introducing me to this very elegant and rich theory of Teichmüller spaces, his cheerful explanations and his kindly comprehension of my backgrounds in mathematics as well as my interests.

Contents

1 Definitions and properties of $\mathcal{T}(S)$

The aim of this section is to introduce the definition of the classical Teichmüller spaces (or $\mathrm{PSL}_2(\mathbb{R})$-Teichmüller space) of a ciliated surface. First, we define ciliated surfaces and study their possible triangulations. The presentation will follow those given in [FG05] and [FG06]. Then, we define the mapping class group of a differentiable surface and give a combinatorial description as a definition of the *modular groupoid* in terms of generators and relations. This part mainly follows [FG06] and [CF99]. The equivalence of the three definitions of the Teichmüller space $\mathcal{T}(S)$ of a surface S is given just as in [FG06]. Then, shear coordinates on the Teichmüller $\mathcal{X}$-space corresponding to a triangulation are constructed according to [FG05]. Eventually, the canonical Poisson structure on the Teichmüller spaces given by the Weil-Peterson form is investigated as well as the links between the two Teichmüller spaces $\mathcal{T}(S)$ and $\mathcal{T}^x(S)$.

1.1 Ciliated surfaces and triangulations

Ciliated surfaces are defined in a way such that they have sufficiently good properties in order to define their Teichmüller spaces. After defining them properly, we study their triangulations, how their topology constraints their triangulations, and recall an important theorem of triangulations of such surfaces.

Ciliated surfaces

Definition 1. *A topological surface S is a topological space such that each point of S has a neighbourhood which is homeomorphic to some open subset of the real plane $\mathbb{R}^2$.*

Compact topological surfaces are classified by their genus. This classification has been initiated by Möbius and Jordan in the middle of the 19th century, but not proven rigorously until Brahana's proof [Bra21] published in 1926. In the sequel, we will consider the following generalisation of topological surfaces:

Definition 2. *A ciliated surface of genus g is an open oriented topological surface with boundary and with a finite set of marked points on the boundary called cilia, constructed from the compact topological surface of the same genus by removing a finite number of disjoint disks and then marking the cilia. Let's denote by $S_{(g,P)}$ (where $P = (p_1, ..., p_s)$) the surface of genus g with s boundary components and p_i cilia on the i-th boundary component. We say that $S_{(g,P)}$ is <u>hyperbolic</u> if one of the following proposition holds:*

- $g = 0$ *and* $s \geq 3$
- $g = 1$ *and* $s \geq 1$
- $g \geq 2$

In the following we will always assume that $S_{(g,P)}$ is hyperbolic, and that $s \geq 1$.

Remark 1. *The fondamental group of $S_{(g,P)}$ is independent of the cilia. The classification of topological surfaces shows that $\pi_1(S_{(g,P)})$ admits $2g$ canonical generators denoted $a_1, b_1, ..., a_g, b_g$ plus s generators denoted by $c_1, ..., c_s$ (the latter corresponding to loops surrounding the boundary components), together with the relation:*

$$[a_1, b_1]...[a_g, b_g] = c_1...c_s$$

Hence:

$$\pi_1(S_{(g,P)}) =< a_1, b_1, ..., a_g, b_g, c_1, ..., c_s | [a_1, b_1]...[a_g, b_g] = c_1...c_s >$$

Triangulations of ciliated surfaces and topological constraints A boundary component without cilia is called a *hole*. A *triangulation* Γ of $S_{(g,P)}$ is a decomposition of $S_{(g,P)}$ into triangles such that any vertex is either a shrunked hole or a cilium. Γ defines a cellular decomposition of $S_{(g,P)}$. An edge of Γ which belongs to two triangles is called an *internal edge*. Otherwise, it's an *external edge*. Let:

- $F(\Gamma)$ be the set of triangles of Γ
- $E(\Gamma)$ be the set of edges of Γ
- $E_0(\Gamma)$ be the set of external edges of Γ
- $V(\Gamma)$ be the set of vertices of Γ

The topology of $S_{(g,P)}$ constraints the cardinality of these sets. Let h be the total number of holes and $c = \sum_{i\in[|1,s|]} p_i$ the total number of cilia in $S_{(g,P)}$. First:

$$\#V(\Gamma) = h + c$$

and as each boundary component of S is homeomorphic to a circle:

$$\#E_0(\Gamma) = c$$

Then, since the Euler characteristic of $S_{(g,P)}$ is given by the two following expressions:

$$\chi(S_{(g,P)}) = \#F(\Gamma) - \#E(\Gamma) + \#V(\Gamma) = 2 - 2g + h - s$$

and since Γ is a triangulation:

$$\#F(\Gamma) = 2\#E(\Gamma) - \#E_0(\Gamma)$$

we have:

$$\#E(\Gamma) = 6g - 6 + 3s + 2c$$

hence the number of internal edges of Γ is completely specified by the topology:

$$\boxed{\#E(\Gamma) - \#E_0(\Gamma) = 6g - 6 + 3s + c}$$

as well as the number of faces:

$$\boxed{\#F(\Gamma) = 4g - 4 + 2s + c}$$

Topologies of triangulations The topology of a triangulation Γ of $S_{(g,P)}$ can be encoded in a $(\#E(\Gamma) \times \#E(\Gamma))$-skew-symmetric matrix ϵ. Let $\alpha, \beta \in E(\Gamma)$ and define the α,β-coefficient of ϵ to be:

$$\epsilon^{\alpha\beta} = \sum_{i\in F(\Gamma)} < \alpha, i, \beta >$$

where $< \alpha, i, \beta >$ equals:

- 1 if α and β are sides of the triangle i and if β is in the clockwise direction from α with respect to their common vertex
- (-1) if α and β are sides of the triangle i and if β is in the couterclockwise direction from α with respect to their common vertex
- 0 otherwise

Remark 2. *Each coefficient in the ϵ-matrix belongs to the set* $\{0, \pm 1, \pm 2\}$.

As we will see later, the choice of a triangulation of a ciliated surface defines a set of coordinates on its Teichmüller space $\mathcal{T}(S_{(g,P)})$. These coordinates *a priori* depend of course on the chosen triangulation. Moreover, one has the following result (see [FG05]):

Remark 3. *The number of triangulations of $S_{(g,P)}$ is infinite unless one of the two following hypotheses holds:*

- $g = 0$ *and* $P = (k)$
- $g = 0$ *and* $P = (k, 0)$

Hence, if Γ and Γ' are two different triangulations of the same ciliated surface $S_{(g,P)}$, one wants to be able to relate the coordinates corresponding to Γ to the ones corresponding to Γ'. The next result tells that two different triangulations of ciliated surfaces are always finitely related. It has been proven by Penner in [Pen87], as well as by Harer in [Har86].

Definition 3. *Let Γ be a triangulation of a ciliated surface, and α an internal edge of Γ. The flip of the edge α corresponds to the modification of Γ depicted in the figure 1.*

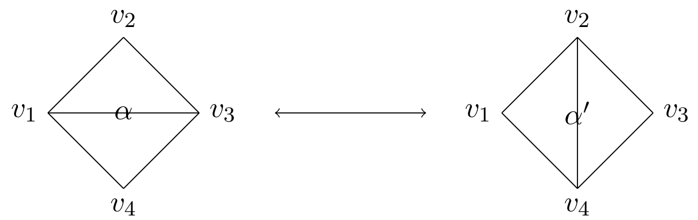


Figure 1: Flip of the internal edge α.

Theorem 1.1. *Let $S_{(g,P)}$ be a ciliated surface and Γ, Γ' two triangulations of $S_{(g,P)}$. Then there exists a finite sequence of flips linking Γ to Γ'.*

Remark 4. *Given two distinct triangulations Γ and Γ' of the same ciliated surface, there is no canonical way to pair the edges of Γ with the edges of Γ'. However, any sequence of flips transforming Γ into Γ' induces such an identification. Indeed, a flip identifies bijectively the set of edges of the original triangulation with the set of edges of the flipped one.*

The next proposition describes the modification of the ϵ-matrix under a flip:

Proposition 1.2. *Let $S_{(g,P)}$ be a ciliated surface and Γ a triangulation of $S_{(g,P)}$. Let γ be an internal edge of Γ. Let Γ_γ be the triangulation obtained by the flip of γ, and for all edge $\alpha \in E(\Gamma)$, let α' be the corresponding edge in $E(\Gamma_\gamma)$. Then:*

- $\epsilon^{\alpha'\beta'} = -\epsilon^{\alpha\beta}$ *if* $\alpha = \gamma$ *or* $\beta = \gamma$
- $\epsilon^{\alpha'\beta'} = \epsilon^{\alpha\beta} + \frac{1}{2}(\epsilon^{\alpha\gamma}|\epsilon^{\gamma\beta}| + |\epsilon^{\alpha\gamma}|\epsilon^{\gamma\beta})$ *otherwise.*

1.2 A combinatorial description of the mapping class group

Let's recall that two differentiable surfaces S and S' are diffeomorphic if and only if they are homeomorphic. The mapping class group (or modular group) of a surface is the group of homotopy classes of diffeomorphisms of S which preserves the orientation and the cilia. Whereas a naive description seems rather difficult, it is possible to describe a category called the modular groupoid in terms of generators and relations. The modular groupoid of a surface is of course very related to the mapping class group of this surface.

Definition 4. *Let S be a ciliated surface. The* <u>mapping class group</u> *(or* <u>modular group</u>*) of S is denoted by $\mathcal{D}(S)$ and is defined by:*

$$\mathcal{D}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$$

where $\mathrm{Diff}(S)$ is the group of orientation-preserving diffeomorphisms of S preserving the set of cilia, and $\mathrm{Diff}_0(S)$ is the connected component of identity in $\mathrm{Diff}(S)$.. Equivalently, $\mathcal{D}(S)$ is the group of homotopy classes of diffeomorphisms of S preserving the orientation and the set of cilia.

The description of the mapping group itself is rather difficult, but it's possible to give a combinatorial description of the a groupoid constructed in a way that every automorphism group is isomorphic to $\mathcal{D}(S)$.

Definition 5. *A groupoid is a category $\mathcal{C}$ in which every morphism is invertible, and such that there exists a morphism between any two objects of $\mathcal{C}$. It implies that for any $A \in \mathrm{Ob}(\mathcal{C})$, $\mathrm{Hom}_{\mathcal{C}}(A, A)$ is a group, and that $\forall A, B \in \mathrm{Ob}(\mathcal{C})$, the groups $\mathrm{Hom}_{\mathcal{C}}(A, A)$ and $\mathrm{Hom}_{\mathcal{C}}(B, B)$ are isomorphic.*

Now we define of the modular groupoid of a ciliated surface. We follow the description given in the appendix A of [FG05]. A slightly but equivalent different introduction using ribbon graphs is given in 3. of [FG06], and in [CF99].

Definition 6. *Let S be some ciliated surface and let $\Gamma(S)$ be the set of isotopy classes of marked triangulations of S. By "marking" of $\Gamma \in \Gamma(S)$ we mean a numeration its edges. The marking prevents the coincidences which can arise from the potentially non-trivial symmetry of the graphs. The mapping class group $\mathcal{D}(S)$ of S acts obviously freely on $\Gamma(S)$. Let:*

$$|\Gamma|(S) = \Gamma(S)/\mathcal{D}(S)$$

be the set of combinatorial types of marked triangulations of S.

Definition 7. *The modular groupoid of S is the groupoid such that:*

- $\mathrm{Ob} = |\Gamma|(S)$
- $\mathrm{Hom}(|\Gamma|, |\Gamma_1|)$ *is the set of equivalence classes of couples (Γ, Γ_1) where Γ and Γ_1 are triangulations of S with respective combinatorial type $|\Gamma|$ and $|\Gamma_1|$. (Γ, Γ_1) and (Γ', Γ'_1) are equivalent if there exists $g \in \mathcal{D}(S)$ such that $(\Gamma, \Gamma_1) = (g \cdot \Gamma', g \cdot \Gamma'_1)$. Let $|\Gamma, \Gamma_1|$ be the element of $\mathrm{Hom}(|\Gamma|, |\Gamma_1|)$ corresponding to the equivalence class of the couple (Γ, Γ_1).*

Remark 5. *The composition is well-defined, $|\Gamma, \Gamma|$ is the identity morphism from $|\Gamma|$ to itself, and for any $|\Gamma| \in |\Gamma|(S)$, $\mathrm{Hom}(|\Gamma|, |\Gamma|) = \mathcal{D}(S)$*

Let Γ be a triangulation of S, and α an internal edge of Γ. Let Γ_α be the triangulation obtained from Γ by the flip of α. If one considers the couple (Γ, Γ_α), its equivalence class is a morphism in $\mathrm{Hom}(|\Gamma|, |\Gamma_\alpha|)$. We still call this morphism a *flip*. The modular groupoid is generated by the flips only, and the flips satisfy three different relations, given in the following proposition (see [FG05]).

Proposition 1.3.

- *(The square of a flip is identity) Let α be an edge of Γ. Then:*

$$|\Gamma_\alpha, \Gamma| \circ |\Gamma, \Gamma_\alpha| = \mathrm{id}_{\Gamma,\Gamma}$$

- *(Flips in* <u>disjoint</u> *edges commute) Let α and β be two edges of Γ sharing no vertex. Then:*

$$|\Gamma_\alpha, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\alpha| = |\Gamma_\beta, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\beta|$$

- *(Flips satisfy the pentagon relation) Let α and β be two edges of Γ having a vertex in common. Then:*

$$|\Gamma_\alpha, \Gamma| \circ |\Gamma_{\beta\alpha}, \Gamma_\alpha| \circ |\Gamma_{\alpha\beta}, \Gamma_{\beta\alpha}| \circ |\Gamma_\beta, \Gamma_{\alpha\beta}| \circ |\Gamma, \Gamma_\beta| = \mathrm{id}_{\Gamma,\Gamma}$$

1.3 Definition of $\mathcal{T}^x(S_{(g,P)})$

The goal of this part is the definition of the Teichmüller space

Teichmüller space of hyperbolic ciliated surface without cilium Let $S_{(g,P)}$ be a hyperbolic ciliated surface. We recall the equivalence between three different possible definitions of the Teichmüller space of $S_{(g,P)}$, following 3.7 of [FG06]. Recall that a riemannian metric of constant curvature -1 is said to be *hyperbolic.*

Proposition 1.4. *There is a one-to-one correspondence between:*

1. *The space of complex structures on $S_{(g,P)}$ modulo* $\mathrm{Diff}^0(S_{(g,P)})$
2. *The space of faithful discrete representations $\pi_1(S_{(g,P)}) \to \mathrm{PSL}_2(\mathbb{R})$ modulo $\mathrm{PSL}_2(\mathbb{R})$-conjugation*
3. *The space of hyperbolic metrics on* $\mathrm{int}(S_{(g,P)})$ *modulo* $\mathrm{Diff}^0(S_{(g,P_0)})$ *such that this hyperbolic structure degenerates at each cilia and with geodesic boundary.*

Proof. The *Riemann uniformisation theorem* states that any simply connected Riemann surface is conformally equivalent to the Riemann sphere, the complex plane or the Poincaré half-plane $\mathbb{H}$. $S_{(g,P_0)}$ being hyperbolic, it can be constructed as the quotient of $\mathbb{H}$ by a discrete subgroup Δ of its automorphisms group $\mathrm{PSL}_2(\mathbb{R})$. As $\mathbb{H}$ is the universal cover of $S_{(g,P_0)}$, Δ is canonically isomorphic to $\pi_1(S_{(g,P_0)})$. Hence the equivalence 1) $\Leftrightarrow$ 2). The canonical hyperbolic metric on $\mathbb{H}$ induces an hyperbolic metric on $S_{(g,P_0)}$, thus we have 2) $\Rightarrow$ 3). Taking the universal cover of $S_{(g,P_0)}$ endowed with a complete hyperbolic metric yields the Poincaré half plane, hence 3) $\Rightarrow$ 2). □

Definition 8. *A discrete and finitely-generated subgroup of* $\mathrm{PSL}_2(\mathbb{R})$ *is called a Fuchsian group. The isomorphism:*

$$S_{(g,P_0)} \to \mathbb{H}/\Delta$$

is called a Fuchsian model for $S_{(g,P_0)}$.

Let's recall an important property of hyperbolic closed riemannian manifolds.

Proposition 1.5. *Let M be a riemannian closed manifold with constant negative curvature. Then every non-trivial free homotopy class of closed curves on M contains exactly one geodesic.*

Its proof of which can be found in [Thu97] (proposition 5.3.1). It implies that there is a unique geodesic surrounding each hole of $S_{(g,P_0)}$, except in the case where a hole degenerates to a puncture.

Classification of $\mathrm{PSL}_2(\mathbb{R})$-elements $\mathrm{PSL}_2(\mathbb{R})$ is isomorphic to the group of automorphisms of $\mathbb{H}$. Let $M \in \mathrm{PSL}_2(\mathbb{R})$ given by:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

It describes the automorphism of $\mathbb{H}$ acting as:

$$M \cdot z = \frac{az+b}{cz+d}$$

The condition for $z_0 \in \mathbb{H}$ to be a stable point under the action of M is:

$$cz_0^2 + (d-a)z_0 - b = 0$$

hence:

$$\lambda_{1,2} = \frac{(a-d) \pm \sqrt{(d-a)^2 - 4bc}}{2c}$$

$$\lambda_{1,2} = \frac{(a-d) \pm \sqrt{(a+d)^2 - 4}}{2c}$$

with a slight abuse of notation if $|\operatorname{Tr}\{M\}| < 2$. However, one can note that:

- if $|\operatorname{Tr}(M)| < 2$, M has two conjugate stable points in $\mathbb{C}$, hence one stable point in $\mathbb{H}$. M is said to be *elliptic*.
- if $|\operatorname{Tr}(M)| = 2$, M has a single real stable point lying on $\partial\mathbb{H} = \mathbb{RP}^1$. M is said to be *parabolic*.
- if $|\operatorname{Tr}(M)| > 2$, M has a two real stable points on $\mathbb{RP}^1$. M is said to be *hyperbolic*.

Hence parabolic and hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ act freely on $\mathbb{H}$, while the action of an elliptic element has a stable point. Thus Δ must only contain parabolic and hyperbolic elements. Let's study the dynamics of parabolic and hyperbolic elements of $\mathrm{PSL}_2(\mathbb{R})$ on $\mathbb{H}$. For simplicity, we will consider the action of the following parabolic element:

$$M_P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

and the action of the following hyperbolic one:

$$M_H = \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

The next two figures show some of the features of the action on $\mathbb{H}$ of two such matrices.

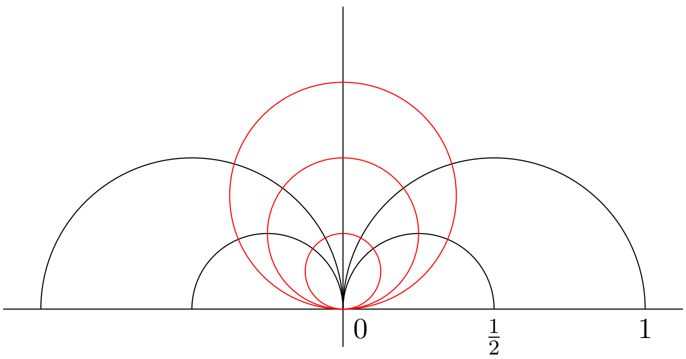


Figure 2: The action of M_P on $\mathbb{H}$ is given by $z \mapsto \frac{z}{z+1}$. Its fixed point is 0. This automorphism of $\mathbb{H}$ sends any geodesics ending at 0 to another geodesic ending at 0. For example, the image of the line $i\mathbb{R}$ is the circle centered at $(0, \frac{1}{2})$ and of radius $\frac{1}{2}$. The stable sets of interest of this action are the horocycles centered at 0, shown in red on the picture. The quotient of $\mathbb{H}$ by this action is conformally equivalent to a punctured disk.

Eventually, let's recall that while an annulus and a punctured disk are homeomorphic, they are not conformally equivalent. Hence one has to distinguish two different conformal types of boundaries without cilium:

- the *holes*, which neighbourhood is conformally equivalent to an annulus
- the *punctures*, which neighbourhood is conformally equivalent to a punctured disk.

Let $S_{(g,P)}$ be a ciliated surface endowed with a hyperbolic riemannian metric. Since homotopy classes of closed curves on $S_{(g,P)}$ are in one-to-one correspondence with the conjugacy classes of the

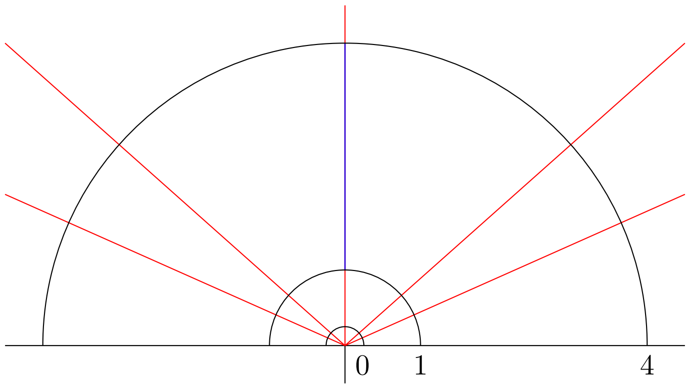


Figure 3: The action of M_H on $\mathbb{H}$ is given by $z \mapsto 4z$. Its fixed points are 0 and ∞. This automorphism sends any half-circle centered at 0 to another half-circle centered at 0. For example, the image of the geodesic with endpoints $\{\pm 1\} \in \mathbb{RP}^1$ is the geodesic with endpoints $\{\pm 4\} \in \mathbb{RP}^1$. The stable sets of interest of this action are the half-lines ending at 0 and lying in $\mathbb{H}$. The quotient of the hyperbolic half-plane by this action is conformally equivalent to an annulus. This annulus can for example be described by choosing the fundamental domain bounded by the circles of radii 1 and 4. The two circles are identified by the action of M_H. The homotopy class of loops surrounding the hole once contains a unique closed geodesic, which is $\{i\lambda | \lambda \in [1,4]\}$ if we keep the same fundamental domain.

fundamental group, a hyperbolic element $\gamma \in \Delta \subset \mathrm{PSL}_2(\mathbb{R})$ corresponds to a unique geodesic on $S_{(g,P)}$. Perhaps after the conjugation by some $\mathrm{PSL}_2(\mathbb{R})$-element, one can suppose that the fixed points of γ are 0 and ∞. According to the discussion below the picture showing the dynamic of a hyperbolic elements in $\mathrm{PSL}_2(\mathbb{R})$, and if λ_1 and λ_2 are the two eigenvalues of γ, then the length of the geodesic is given by the simple formula:

$$l(\gamma) = |\ln\left(\frac{\lambda_1}{\lambda_2}\right)|$$

which is invariant by conjugation, and hence does not depend on the chosen representation of $\pi_1(S_{(g,P)})$. The right-hand side of this formula is well defined even for parabolic elements, defining the length of the "geodesic" surrounding a puncture to be 0, as:

$$\inf_{\alpha \in C} l(\alpha) = 0$$

C being the free homotopy class of closed curves surrounding the puncture.

Definition 9. *Let $S_{(g,P)}$ be a ciliated surface together with a Fuchsian model with the Fuchsian group Δ, and let $\pi : \mathbb{H} \to \mathbb{H}/\Delta \simeq S_{(g,P)}$ be the corresponding projection. Let C be the set of all π-preimages of the cilia, and let $\mathrm{St} \subset /mathbbH$ be the set of the points which are stable for at least an $M \in \Delta$. Then Δ preserves the set $C \cup \mathrm{St}$ and also its convex hull in $\mathbb{H}$. Δ acts freely on this convex hull (it is a subset of $\mathbb{H}$). The convex core of $S_{(g,P)}$ is the quotient of this convex hull by Δ.*

The convex core of $S_{(g,P)}$ endowed with a hyperbolic metric may also be obtained in the following way. Start with $S_{(g,P)}$. For any hole of $S_{(g,P)}$ consider the unique geodesic contained in the free homotopy class of closed curves homotopic to it, and remove the annulus which is between this geodesic and the boundary component. For any pair of cilia which are adjacent on the same boundary component, and for any connected segment of the boundary minus the cilia connecting them, take the geodesics going from one cilia to another and homotopic to this segment, and remove the part of $S_{(g,P)}$ which is between this geodesic and the corresponding segment. What remains is the convex core of $S_{(g,P)}$.

Teichmüller $\mathcal{X}$-space of a ciliated surface with holes

Definition 10. *Let $S_{(g,P)}$ be a ciliated surface and $\mathcal{O}$ its orientation. The Teichmüller $\mathcal{X}$-space $\mathcal{T}^x(S)$ of a ciliated surface with holes $S_{(g,P)}$ is the space of hyperbolic structures on S with geodesic boundary, modulo* $\mathrm{Diff}^0(S)$*, together with the orientation of all holes but the punctures, and such that the π-preimages of all cilia are in $\partial\mathbb{H} = \mathbb{RP}^1$. If ρ is a boundary component of S and if S is endowed with a complex structure for which ρ is a hole, let $\mathcal{O}_\rho$ be the orientation of ρ. We write $\mathcal{O} = \mathcal{O}_\rho$ is the two orientations agrees, and $\mathcal{O} \neq \mathcal{O}_\rho$ if they disagrees.*

The orientation of holes is needed for technical reasons which will hopefully become clear in a while. For each boundary component ρ of $S_{(g,P)}$, we will construct a function:

$$r^\rho : \mathcal{T}^x(S) \to \mathbb{R}_>0$$

which is given by:

$$r^\rho(x \in \mathcal{T}^x(S)) = \begin{cases} e^{l(\gamma)} \text{ if } \mathcal{O} = \mathcal{O}_\rho \\ e^{-l(\gamma)} \text{ if } \mathcal{O} \neq \mathcal{O}_\rho \\ 1 \text{ if } \rho \text{ is a puncture} \end{cases}$$

1.4 Construction of the shear coordinates on $\mathcal{T}^x(S_{(g,P)})$

We are going to explain how one can construct shear coordinates on $\mathcal{T}^x(S)$, and recall how from shear coordinates on can reconstruct the complex structure of a hyperbolic ciliated surface.

Construction of the coordinates associated with a complex structure on $S_{(g,P)}$ Suppose that $S_{(g,P)}$ is endowed with a complex structure. Since $S_{(g,P)}$ is hyperbolic, it inherits a riemannian hyperbolic structure, hence any edge of the triangulation whose ends are two punctures, two cilia, or a puncture and a cilium, can be made geodesic in a canonical way. However, the holes behave as punctures from the triangulation point of view (i.e the topological point of view), but holes and punctures are distinguished from the complex structure. One wants a triangulation of $S_{(g,P)}$ with geodesic edges of infinite length, in order to parametrise $\mathcal{T}^x(S_{(g,P)})$ with the gluing data of hyperbolic ideal triangles only. Hence one has to specify the orientation of holes. For the procedure we are about to describe, the one induced by $\mathcal{O}$ is not canonical in any way.

Consider the case of an edge of the triangulation whose ends are a puncture and a hole. Consider the unique geodesic loop homotopic to the boundary component corresponding to the latter, and chose some point on it. There exists a unique geodesic connecting the puncture and that point. The orientation of the hole induces an orientation of the geodesic loop around it. Move the point on the latter in the direction given by the orientation of this loop. The desired geodesic is the limiting one of the process. One can uniquely deform an edge linking two holes in the same way. This construction does not change the isotopy class of the triangulation.

Eventually one has a triangulation of the convex core of $S_{(g,P)}$ with geodesic edges of infinite length.

Lift this triangulation to the universal cover $\mathbb{H}$ of $S_{(g,P)}$, and consider an edge α between two adjacent triangles. The latter form an ideal hyperbolic quadrilateral. The quadruple of its vertices has an invariant under the action of $\mathrm{PSL}_2(\mathbb{R})$, which can be chosen to be minus the cross-ratio of these four points: up to $\mathrm{PSL}_2(\mathbb{R})$-conjugation, one of the two triangles of which α is a side can be sent to the ideal hyperbolic triangle of vertices -1,0 and ∞. Then the invariant is given by the coordinate of the fourth point, which is a strictly positive real number. If one had chosen the other triangle to be send to $\{-1, 0, \infty\}$, the coordinate of the fourth point would have been the same.

To each internal edge of the triangulation one assigns this strictly positive real number. Hence this construction defines a map:

$$\mathcal{T}^x(S_{(g,P)}) \to \mathbb{R}_{>}^{6g-6+3s+c}$$

Now one has to show that this map is one-to-one.

Construction of a complex structure on $S_{(g,P)}$ from the coordinates Following [FG05], we are going to describe two equivalent ways to reconstruct a ciliated complex structure starting from a triangulation of $S_{(g,P)}$ with a real positive number assigned to each internal edge.

Since the triangulation of $S_{(g,P)}$ has vertices corresponding to holes, punctures or cilia, each triangle is homeomorphic to an ideal hyperbolic triangle in $\mathbb{H}$, where by ideal we mean with geodesic edges and with vertices in $\partial\mathbb{H} = \mathbb{RP}^1$. Let's assign to each triangle the complex structure corresponding to this ideal hyperbolic triangle. In the next section we will see that two ideal hyperbolic triangles are always isomorphic. We will also see that one can parametrise the gluing of two ideal hyperbolic triangles by a strictly positive real parameter (which is the cross-ratio of the four vertices). Thus the strictly positive real numbers on the edges are the gluing data one needs in order to know how unambiguously glue two adjacent triangles.

The second way consists of constructing a Fuchsian group Δ in $\mathrm{PSL}_2(\mathbb{R})$ corresponding to the complex structure given by the numbers on the edges of the triangulation. First construct the dual graph to the triangulation. To any edge of this dual graph corresponding to an edge of the initial triangulation carrying the value x, assign the matrix:

$$B(x) = \begin{pmatrix} 0 & \sqrt{x} \\ \frac{-1}{\sqrt{x}} & 0 \end{pmatrix}$$

Now, one replaces each vertex in the new graph (hence corresponding to a face in the initial triangulation) by a triangle with clockwise oriented edges. This replacement is presented in the following figure.

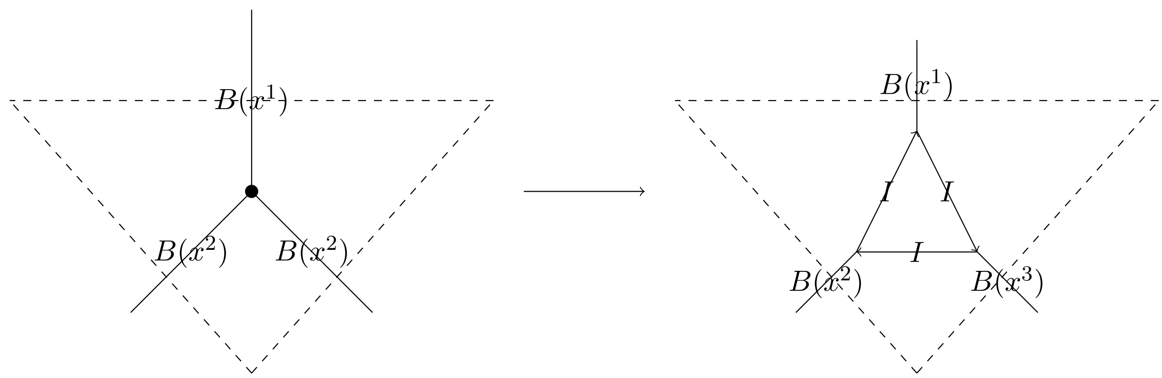


Figure 4: Blow-up of the vertices of the dual graph. The initial triangulation is shown in dashed lines.

The matrix:

$$I = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is assigned to each new edge which has appeared during this process. Let $\gamma \in \pi_1(S_{(g,P)})$. γ is homotopic to an oriented path on this new graph. One assigns an element of $\mathrm{PSL}_2(\mathbb{R})$ to γ by the following recipe. One chooses a vertex on the path, and follows the path corresponding to γ in the direction prescribed by the orientation. The matrices assigned to the edges of the path are multiplied in the order in which one meets them, taking the inverse of the matrix each time the orientation of the path disagrees with the orientation of the edge.

Remark 6. *$B(x) = B(x)^{-1}$ in $\mathrm{PSL}_2(\mathbb{R})$.*

Remark 7. *It is interesting to see how I an $B(x)$ act on the ideal hyperbolic triangle of vertices ∞,0 and -1. I is the matrix which sends ∞ to -1, -1 to 0 and 0 to ∞. It corresponds to a counterclockwise rotation of the triangle $(\infty, -1, 0)$. If $x \in \mathbb{R}_>$, the matrix $B(x)$ sends ∞ to 0, -1 to x and 0 to ∞.*

Proposition 1.6. *Let $\alpha \in \pi_1(S_{(g,P)})$ and M the corresponding matrix given by this construction. Then M is either parabolic or hyperbolic.*

Proof. By construction, M can be written as:

$$M = \prod_{i=1}^{n} B(x_i) I^{\omega_i}$$

where $\omega_i = \pm 1$. Note that:

$$B(x)I = -\begin{pmatrix} \sqrt{x} & 0 \\ \frac{1}{\sqrt{x}} & \frac{1}{\sqrt{x}} \end{pmatrix}$$

hence $B(x)I$ is lower-triangular. Similarly:

$$B(x)I^{-1} = -\begin{pmatrix} \sqrt{x} & \sqrt{x} \\ 0 & \frac{1}{\sqrt{x}} \end{pmatrix}$$

hence $B(x)I$ is upper-triangular. Thus:

$$M = (-1)^n \prod_{i=1}^{n} C_i(x_i)$$

where the $C_i(x_i)$ are matrices in $\mathrm{PSL}_2(\mathbb{R})$ with positive coefficients. Therefore any product of C_i is a matrix in $\mathrm{PSL}_2(\mathbb{R})$ with positive coefficients and it implies that its diagonal coefficients are non-zero. Note that a product of two C_i is always either parabolic or hyperbolic. For $a, c \in \mathbb{R}_>$ and $b, d, e \in \mathbb{R}_+$:

$$\mathrm{Tr}\left(\begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix}\begin{pmatrix} c & d \\ e & \frac{1+de}{c} \end{pmatrix}\right) = ac + \frac{1}{ac} + be + \frac{de}{ac} \geq 2$$

hence the proposition follows by recurrence. ∎

Proposition 1.7. *Let $\alpha \in \pi_1(S_{(g,P)})$ be a loop homotopic to a boundary component without cilia $\rho \subset \partial S_{(g,P)}$. Then the corresponding matrix in $\mathrm{PSL}_2(\mathbb{R})$ is either upper-triangular or lower-triangular, and its diagonal coefficients are:*

$$(-1)^n \begin{pmatrix} \sqrt{\prod_{i=1}^{n} x_i} & * \\ * & \frac{1}{\sqrt{\prod_{i=1}^{n} x_i}} \end{pmatrix}$$

where n is the number of edges which are incident to the vertex corresponding to ρ, and where x_i is the value carried by the i-th edge. The geodesic surrounding the hole has length $|\ln(\prod_{i=1}^{n} x_i)|$. The map:

$$r^\rho : \mathcal{T}(S_{(g,P)}) \to \mathbb{R}_+$$

assigns the product $\prod_{i=1}^{n} x_i$ to any complex structure, hence if r^ρ is strictly bigger that 1 the orientation of the hole coincides with the orientation of $S_{(g,P)}$, if it is strictly lower then the two orientations differ, and if it is equal to 1 then ρ is in fact a puncture.

Change of coordinates under a flip These shear coordinates on $\mathcal{T}(S_{(g,P)})$ obviously depend on the triangulation. The change of coordinates corresponding to a change of the triangulation can always be computed thanks to the following formula giving the change of coordinates under a flip.

Proposition 1.8. *Let α be an internal edge of a triangulation Γ of $S_{(g,P)}$, let Γ_α be the triangulation obtained by the flip of α. Let $\{x_\alpha\}$ be the set of shear coordinates corresponding to Γ, and $\{x'_\alpha\}$ the set of shear coordinates corresponding to Γ', with the natural identification of edges between Γ and Γ_α. Then:*

$$x^{\beta'} = \begin{cases} (x^\alpha)^{-1} \text{ if } \beta = \alpha \\ x^\beta(1+x^\alpha)^{\epsilon^{\alpha\beta}} \text{ if } \epsilon^{\alpha\beta} \geq 0 \\ x^\beta(1+(x^\alpha)^{-1})^{\epsilon^{\alpha\beta}} \text{ if } \epsilon^{\alpha\beta} \leq 0 \end{cases}$$

Proof. These formula can be directly computed using a Fuchsian model of $S_{(g,P)}$, but in the next section we will discuss different approaches of the cross-ratio which will make their computation very easy. □

An example: shear coordinates on the Teichmüller space of the torus with one boundary component Let's consider a triangulation of the torus with one boundary component ρ without cilia.

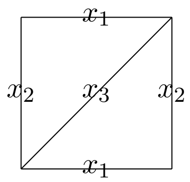


Figure 5: Triangulation of the torus with one puncture

Any hyperbolic structure on this surface can be described by a triple of strictly positive real numbers. The length of the geodesic surrounding the boundary component is:

$$l = |\ln(x_1^2 x_2^2 x_3^2)|$$

Now consider the hyperbolic structures for which $x_1 = x_2 = x_3 =: x$ described by x. The length of the geodesics corresponding to the conventional generators of $\pi_1(\mathbb{T}\backslash\{\text{pt}\})$ is:

$$l(x) = \ln\left(\frac{T(x) + \sqrt{D(x)}}{T(x) - \sqrt{D(x)}}\right)$$

where:

$$T(x) = \frac{1}{x^2} + \frac{2}{x} - 1 + 2x + x^2,\ D(x) = \frac{1}{x} + 1 + x$$

1.5 The Weil-Peterson form on $\mathcal{T}(S_{(g,P)})$

Teichmüller spaces of ciliated surfaces interestingly have a lot of structure. $\mathcal{T}^x(S)$ is a real manifold, and $\mathcal{T}(S)$ is real manifold with corners. Moreover, they can be endowed with a Poisson structure, which can be quantised. We will not develop the last point, but recall some facts about Poisson brackets on real manifolds, and study the link between $\mathcal{T}^x(S)$ and $\mathcal{T}(S)$.

Poisson structures

Definition 11. *A Poisson bracket on a smooth manifold M is a bilinear map:*

$$\{.,.\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$$

satisfying the following properties for all $f, g, h \in \mathcal{C}^\infty(M)$:

1. $\{f, g\} = -\{g, f\}$ *(skew-symmetry)*
2. $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$ *(Jacobi identity)*
3. $\{fg, h\} = f\{g, h\} + g\{f, h\}$ *(Leibniz's rule)*

A smooth manifold endowed with a Poisson bracket will be called a Poisson manifold.

Definition 12. *A smooth map $\psi : M \to N$ between two Poisson manifolds will be called a Poisson map if it preserves the Poisson structures, meaning that for all $f_1, f_2 \in \mathcal{C}^\infty(N)$:*

$$\{f_1, f_2\}_N \circ \psi = \{f_1 \circ \psi, f_2 \circ \psi\}_M$$

Definition 13. *Since a Poisson structure is antisymmetric and a derivation is each of its arguments it can be given by a bivector field, i.e a section of $\bigwedge^2 \mathrm{TM}$. This bivector field π is given by:*

$$\pi(df, dg) := \{f, g\}$$

Suppose that M is simply connected. π defines a map:

$$\pi^\# : \begin{cases} \mathrm{T^*M} & \to & \mathrm{TM} \\ df & \mapsto & \pi(df, d.) \end{cases}$$

Note that an additional property is required for a bivector field to induce a Poisson structure. It corresponds to the Jacobi identity for the Poisson bracket.

Proposition 1.9. *Suppose that M is simply connected, and that it is endowed with a Poisson bracket defining the bivector π. Then the distribution $\operatorname{Im}\pi^\#$ is involutive.*

Proof. Let $V_f \in \operatorname{Im}\pi$ be the vector field $\pi(df, d.) : \mathcal{C}^\infty(M) \to \mathcal{C}^\infty(M)$. Let $V_f V_g \in \operatorname{Im}\pi$ and let $h \in \mathcal{C}^\infty(M)$. Then:

$$[V_f, V_g]h = \pi(df, d\pi(dg, dh)) - \pi(dg, d\pi(df, dh)) = \pi(dh, d\pi(df, dg)) = V_h(\pi(df, dg))$$

because of Jacobi's identity. □

Thus by Frobenius' theorem $\operatorname{Im}\pi^\#$ is integrable. Let S be the foliation of M such that:

$$\mathrm{TS} = (\operatorname{Im}\pi^\#) \subset \mathrm{TM}$$

The bivector field π on M induces a bivector field π_S on S, and for all $\alpha \in T^*_s M$, $\pi_S(\alpha_{|S}) = \pi^\#(\alpha)$ hence the Poisson bracket on S induced by the one on M is non-degenerate.

The canonical Poisson structure on Teichmüller spaces According to [FG06], the space of homomorphisms of the fundamental group of a surface S into any reductive group considered up to conjugation has a canonical Poisson structure.

Definition 14. *The parametrisation of $\mathcal{T}^x(S_{(g,P)})$ described in the previous subsections endows it with a structure of differentiable manifold. It admits a canonical Poisson structure, which is known as the Weil-Peterson Poisson bracket. In shear coordinates, it is given by:*

$$\pi_{WP} = \sum_{\alpha,\beta\in E(\Gamma)} \epsilon^{\alpha\beta} x^\alpha x^\beta \frac{\partial}{\partial x^\alpha} \otimes \frac{\partial}{\partial x^\beta}$$

where ϵ is the matrix corresponding to some triangulation Γ of $S_{(g,P)}$. In fact, this Poisson structure is independent of the triangulation (it can be checked directly that it is invariant under any flip).

Definition 15. *A real manifold of dimension n with corners is a manifold which is locally modelled on:*

$$[0,+\infty[^k \times \mathbb{R}^{n-k}$$

By construction, $\mathcal{T}^x(S_{(g,P)})$ is 2^n-ramified cover of $\mathcal{T}(S_{(g,P)})$, where the projection corresponds to forgetting the orientation of the boundary components. This covering is ramified above the complex structures on $(S_{(g,P)})$ where at least one boundary component is a puncture rather than a hole. Take for example a point $x \in \mathcal{T}(S_{(g,P)})$ and consider some boundary component ρ. If ρ is a hole then above x in $\mathcal{T}^x(S_{(g,P)})$ there are two points x_1 and x_2 which only differ by the orientation of ρ. Take some triangulation Γ of $S_{(g,P)}$ and a numeration of its n edges. Let $H \subset [|1,n|]$ be the subset of the edges incident to ρ, and let x_1^i be the coordinate on the i-th edge for x_1 and x_2^i the same coordinate for x_2. Then we have for example:

$$\prod_{i\in H} x_1^i > 1$$

and

$$\prod_{i\in H} x_2^i < 1$$

If ρ is not a hole but a puncture, x_1 and x_2 coincide and:

$$\prod_{i\in H} x_1^i = \prod_{i\in H} x_2^i = 1$$

Hence corners in $\mathcal{T}(S_{(g,P)})$ corresponds to complex structures on $S_{(g,P)}$ with at least one puncture. If $S_{(g,P)}$ has h holes ρ_i, $i \in [|1,h|]$ and if $\forall i \in [|1,n|]$ one sets:

$$y^i := \ln(x^i)$$

one can choose a parametrisation of $S_{(g,P)}$ of the form:

$$(\sum_{i\in H_1} y^i, ..., \sum_{i\in H_h} y^i, y_{h+1}, ..., y_n)$$

which explicitly gives:

$$\mathcal{T}(S_{(g,P)}) \simeq [0,+\infty[^h \times \mathbb{R}^{n-h}$$

Furthermore, a direct calculation shows that for all hole ρ and for all $k \in [|1,n|]$:

$$\{\prod_{i\in H} x_i, x_k\} = 0$$

Thus the Poisson structure on $\mathcal{T}^x(S_{(g,P)})$ descends to a well-defined Poisson structure on $\mathcal{T}(S_{(g,P)})$.

The Poisson manifold $\mathcal{T}S$ can be quantised in an equivariant way with respect to the mapping class group of S. For that we refer to [CF99].

2 Cross-ratio as moduli spaces of graph $\mathbb{R}^*$-connections

In this section we introduce four different definitions of the cross-ratio of four points in $\mathbb{RP}^1$. The last one presents the cross ratio as the holonomy of graph connections, and can be directly extended to a definition of coordinates on the Teichmüller space of a ciliated surface.

2.1 The cross-ratio of four points in $\mathbb{RP}^1$

The three first definitions of the cross-ratio are rather usual. However, and even if the links between these three approaches are interesting, the aim of this subsection is mainly to understand how this invariant of lines is constructed. In the first approach, one looks at an invariant of a quadruple of points in $\mathbb{RP}^1$. Conversely, in the second approach, the cross-ratio is obtained as an invariant of orbits of quadruples of points in $\mathbb{R}^2$ under the action of $\mathrm{SL}_2(\mathbb{R})$, and this invariant descends to an invariant of four points in $\mathbb{RP}^1$. In the third approach, an auxiliary affine line is needed and one can prove that the cross-ratio does not depend on the choice of this line, which is equivalent to say that it defines an invariant of four points in $\mathbb{RP}^1$. These definitions are given in an order which allows quick proofs of their equivalence.

Group-theory approach The standard representation of the group $\mathrm{SL}_2(\mathbb{R})$ as 2×2 real matrices with unit determinant acts on $\mathbb{R}^2$ via:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

It induces an action of $\mathrm{SL}_2(\mathbb{R})$ on the real projective line $\mathbb{RP}^1$ given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

where $z := \mathbb{R} \cdot \begin{pmatrix} z \\ 1 \end{pmatrix}$.

This action factors through the quotient $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\pm\mathrm{Id}$. This action is transitive, and is if fact 3-transitive in the following sense:

Proposition 2.1. *Given a triple $(z_1, z_2, z_3) \in (\mathbb{RP}^1)^3$ such that the points are pairwise distinct, and such that the standard orientation of the triple coincides with the canonical orientation of $\mathbb{RP}^1$, there exist a unique matrix $M \in \mathrm{PSL}_2(\mathbb{R})$ sending z_1 to ∞, z_2 to -1 and z_3 to 0.*

Proof. First note that the matrix:

$$\pm \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

is the unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ which sends ∞ to -1, -1 to 0 and 0 to ∞. The orientation of the triple (z_1, z_2, z_3) coincides with the standard one on $\mathbb{RP}^1$, thus one can suppose that $z_1 < z_2 < z_3$. Solving the equations corresponding to : $z_1 \to \infty$, $z_2 \to -1$ and $z_3 \to 0$ gives:

$$M = \begin{pmatrix} a & -az_3 \\ a\frac{z_{32}}{z_{21}} & -az_1\frac{z_{32}}{z_{21}} \end{pmatrix}$$

and $\det(M) = 1$ implies:

$$a^2 = \frac{z_{21}}{z_{31}z_{32}}$$

Thus M is uniquely defined in $\mathrm{PSL}_2(\mathbb{R})$. □

Proposition 2.2. *Any matrix $M \in \mathrm{PSL}_2(\mathbb{R})$ preserves the orientation of the triples of points in $\mathbb{RP}^1$.*

Proof. It is impossible to construct a matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending the triple $(\infty, -1, 0)$ to $(0, -1, \infty)$. $\square$

Definition 16. *The triple (z_1, z_2, z_3) is said to be positively oriented if its natural orientation coincides with the standard one of $\mathbb{RP}^1$, else it is said negatively oriented.*

Definition 17. *The space of $\mathrm{PSL}_2(\mathbb{R})$-orbits under the diagonal action on n-uples of distinct points is called the configuration space of n distinct points in $\mathbb{RP}^1$, and denoted $\mathrm{Conf}_n^*(\mathbb{RP}^1)$. The space of $\mathrm{PSL}_2(\mathbb{R})$-orbits of n-uples of points (not necessarily distincts) is called the configuration space of n points in $\mathbb{RP}^1$, and denoted $\mathrm{Conf}_n(\mathbb{RP}^1)$.*

Example 1.

1. *Since for any two points $z_1 \neq z_2 \in \mathbb{RP}^1$ there exists matrices in $\mathrm{PSL}_2(\mathbb{R})$ sending z_1 to ∞ and z_2 to 0, we have that:*

$$\mathrm{Conf}_2^*(\mathbb{RP}^1) = \{\mathrm{pt}\}$$

2. *If (z_1, z_2) is a pair of points in $\mathbb{RP}^1$, they are either distinct either equal. Hence:*

$$\mathrm{Conf}_2(\mathbb{RP}^1) = \{\mathrm{id.}, \mathrm{dist.}\}$$

where id. is the orbit of the couples of distinct points and dist. the orbit of the pairs of distinct points.

3. *Given three distinct points $z_1, z_2, z_3 \in \mathbb{RP}^1$, if the triple (z_1, z_2, z_3) is positively oriented, there exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(\infty, -1, 0)$. If it is negatively oriented, there exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(0, -1, \infty)$. Since there does not exist any matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending $(\infty, -1, 0)$ to $(0, -1, \infty)$, there are two $\mathrm{PSL}_2(\mathbb{R})$-orbits. Hence $\mathrm{Conf}_3^*(\mathbb{RP}^1)$ contains two distinct elements.*

4. *Let (z_1, z_2, z_3) be any triple of points in $\mathbb{RP}^1$. If they are all distincts there are two orbits under $\mathrm{PSL}_2(\mathbb{R})$. There is three orbits for the cases where two points are equal to each other but not to the last one, and a last orbit for the case of three equal points. Thus $\mathrm{Conf}_3(\mathbb{RP}^1)$ contains six points.*

Theorem 2.3. *Let $n > 3$.*

$$\mathrm{Conf}_n^*(\mathbb{RP}^1) \simeq (\mathbb{R}^{n-3} \backslash F) \amalg (\mathbb{R}^{n-3} \backslash F)$$

where F is the subset of $\mathbb{R}^{n-3} = \{(x^1, ..., x^{n-3})\}$ defined by the equations:

$$\begin{cases} \exists i \in [|1, n-3|],\ x^i = 0 \\ \text{or} \\ \exists i \in [|1, n-3|],\ x^i = -1 \\ \text{or} \\ \exists i \neq j \in [|1, n-3|],\ x^i = x^j \end{cases}$$

Hence $\mathrm{Conf}_n^*(\mathbb{RP}^1)$ is an algebraic variety over $\mathbb{R}$.

Proof. Consider the n-uple $(z_1, ..., z_n) \in (\mathbb{RP}^1)^n$. There exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending the triple (z_1, z_2, z_3) to $(\infty, -1, 0)$ or $(0, -1, \infty)$ depending on the orientation of (z_1, z_2, z_3). The value of the other points are free parameters, but cannot be in F since the standard representation of $\mathrm{PSL}_2(\mathbb{R})$ is faithful. $\square$

We're especially interested in the case $n = 4$, where:

$$\mathrm{Conf}_4^*(\mathbb{RP}^1) = (\mathbb{R}\backslash\{-1,0\})_+ \amalg (\mathbb{R}\backslash\{-1,0\})_-$$

With those tools we can define the algebraic cross-ratio of four distinct points in $\mathbb{RP}^1$.

Definition 18. *Let (z_1, z_2, z_3, z_4) be a quadruple of distinct points in $\mathbb{RP}^1$.*

- *If the orientation of the triple (z_1, z_2, z_3) is positive, there exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(\infty, -1, 0)$. This matrix sends z_4 to some point $z \in \mathbb{R}\backslash\{-1,0\}$. The cross-ratio of (z_1, z_2, z_3, z_4) is defined by:*
$$r(z_1, z_2, z_3, z_4) = -z$$
- *If the orientation of the triple (z_1, z_2, z_3) is negative, there exists a unique matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(0, -1, \infty)$. This matrix sends z_4 to some point $z \in \mathbb{R}\backslash\{-1,0\}$. The cross-ratio of (z_1, z_2, z_3, z_4) is defined by:*
$$r(z_1, z_2, z_3, z_4) = -\frac{1}{z}$$
The reason to chose this definition for quadruples where the first triple is negatively oriented will become clearer in the following.

If (z_1, z_2, z_3, z_4) is a triple such that (z_1, z_2, z_3) and (z_3, z_4, z_1) are positively oriented, while (z_1, z_2, z_3) is send to $(\infty, -1, 0)$, z_4 is send to a strictly positive real number. Taking minus the cross-ratio of the four points as parameter, we have:

$$\mathrm{Conf}_4^>(\mathbb{RP}^1) \simeq \mathbb{R}_+^*$$

Geometric approach

Definition 19. *Let D_1, D_2, D_3, D_4 be four distinct lines passing through the origin in $\mathbb{R}^2$. For each line D_i, let $\vec{e_i} \in D_i$ be a non-zero vector: $(\vec{e_i})$ is a basis of the one-dimensional vector-space D_i. The cross-ratio of the quadruple (D_1, D_2, D_3, D_4) is defined to be:*

$$r(D_1, D_2, D_3, D_4) = \frac{\det(\vec{e_1}, \vec{e_2}) \det(\vec{e_3}, \vec{e_4})}{\det(\vec{e_2}, \vec{e_3}) \det(\vec{e_4}, \vec{e_1})}$$

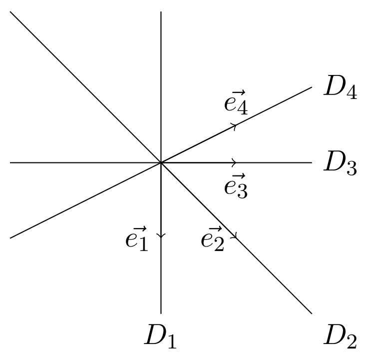


Figure 6: Four lines passing through the origin in $\mathbb{R}^2$, and choice of a basis for each of them

In this definition of the cross-ratio, invariance under the action of $\mathrm{PSL}_2(\mathbb{R})$ is explicit since $\mathrm{PSL}_2(\mathbb{R})$ preserves the determinant. Moreover the determinant is bilinear hence we have the following theorem.

Proposition 2.4. *The cross-ratio of* (D_1, D_2, D_3, D_4) *is well defined on each orbit of quadruple of points under the action of* $(\mathbb{R}^*)^4$.

If one identifies the lines $D_i \in \mathbb{R}^2$ as points $z_i \in \mathbb{RP}^1$, the proposition implies that the cross-ratio of the lines corresponding to z_1, z_2, z_3, z_4 is invariant under the group $\mathrm{PSL}_2(\mathbb{R})$ of projective transformations of $\mathbb{RP}^1$.

- If the triple (z_1, z_2, z_3) is positively oriented there exists a matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(\infty, -1, 0)$. Let z be the point to which is sent z_4. We also refer to the lines which slopes are $\infty, -1, 0, z$ by these numbers. By slope of a line passing through the origin we mean the following: if $\begin{pmatrix} x \\ y \end{pmatrix}$ is a non-zero vector of the line, the slope is the ratio $\frac{x}{y}$. If $y = 0$, the slope is said to be infinite. It can coincide with the intuitive notion of slope provided a good choice of axis in the plane. Let's choose non-zero vectors of each line:

$$\vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{e_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \vec{e_3} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \vec{e_4} = \begin{pmatrix} z \\ 1 \end{pmatrix}$$

  Computing the cross-ratio as a quotient of products of determinants, one has:

$$r(z_1, z_2, z_3, z_4) = \frac{1 \times (-z)}{(-1) \times (-1)} = -z$$

- If the triple (z_1, z_2, z_3) is negatively oriented there exists a matrix in $\mathrm{PSL}_2(\mathbb{R})$ sending it to $(0, -1, \infty)$. Let z be the point to which is sent z_4. Let's choose one non-zero vector in each line:

$$\vec{e_1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \vec{e_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \ \vec{e_3} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{e_4} = \begin{pmatrix} z \\ 1 \end{pmatrix}$$

  Computing the cross-ratio as a quotient of products of determinants, one has:

$$r(z_1, z_2, z_3, z_4) = \frac{1 \times 1}{(-1) \times z} = -\frac{1}{z}$$

Hence we have proven that the two definitions of the cross-ratio we have already presented coincide. Moreover, the second approach shed some light on the reason why we had to take $-\frac{1}{z}$ for the cross-ratio of a quadruple in which the first triple is negatively oriented.

Projective geometry approach There is another way of computing the cross-ratio which is very close to the second definition we gave, and closer to the "historical" definition. Again (D_1, D_2, D_3, D_4) is a quadruple of distinct lines passing through the origin in $\mathbb{R}^2$. To compute the cross-ratio $r(D_1, D_2, D_3, D_4)$, one has to choose an affine line in $\mathbb{R}^2$. In the generic case, its slope is different from the slope of each D_i hence it intersects all of them.

Definition 20. *The cross-ratio of* (D_1, D_2, D_3, D_4) *is defined by:*

$$r(D_1, D_2, D_3, D_4) = \frac{\overline{AB} \cdot \overline{CD}}{\overline{AD} \cdot \overline{CB}}$$

where the notation refers to the figure below.

Proposition 2.5. *This definition of the cross-ratio does not depend on the choice of the affine line. Moreover this definition coincides with the two first definitions.*

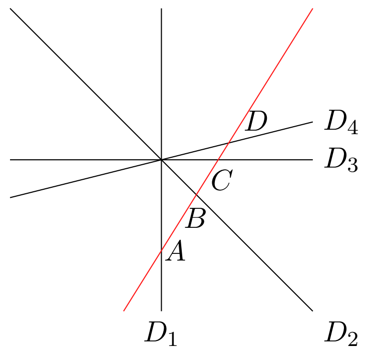


Figure 7: The cross-ratio with the help of an external affine line. Letters refer to intersection points between the affine line and the D_i.

Proof. Let 1, 2, 3, 4 be four lines in $\mathbb{R}^2$, and let:

$$\vec{e_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in 1,\ \vec{e_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in 2,\ \vec{e_3} = \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} \in 3,\ \vec{e_4} = \begin{pmatrix} x_4 \\ y_4 \end{pmatrix} \in 4$$

be nonzero vectors. Let $y = ax + b$ be the equation of a line like the one in red in the figure. For the time being, a is chosen to be different from the slopes of the lines for which we want to compute the cross-ratio. Moreover, b has to be nonzero. Computing the coordinates of the intersection points gives:

$$A = \begin{pmatrix} \frac{x_1 b}{y_1 - a x_1} \\ \frac{b y_1}{y_1 - a x_1} \end{pmatrix},\ B = \begin{pmatrix} \frac{x_2 b}{y_2 - a x_2} \\ \frac{b y_2}{y_2 - a x_2} \end{pmatrix},\ C = \begin{pmatrix} \frac{x_3 b}{y_3 - a x_3} \\ \frac{b y_3}{y_3 - a x_3} \end{pmatrix},\ D = \begin{pmatrix} \frac{x_b}{y_4 - a x_4} \\ \frac{b y_4}{y_4 - a x_4} \end{pmatrix}$$

that is:

$$A = \frac{b}{y_1 - a x_1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix},\ B = \frac{b}{y_2 - a x_2} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix},\ C = \frac{b}{y_3 - a x_3} \begin{pmatrix} x_3 \\ y_3 \end{pmatrix},\ D = \frac{b}{y_4 - a x_4} \begin{pmatrix} x_4 \\ y_4 \end{pmatrix}$$

and thus:

$$r(D_1, D_2, D_3, D_4) = \frac{(x_2 y_1 - x_1 y_2)(x_4 y_3 - x_3 y_4)}{(x_4 y_1 - x_1 y_4)(x_2 y_3 - x_3 y_2)}$$

$$r(D_1, D_2, D_3, D_4) = \frac{\det(\vec{e_1}, \vec{e_2}) \det(\vec{e_3}, \vec{e_4})}{\det(\vec{e_2}, \vec{e_3}) \det(\vec{e_4}, \vec{e_1})}$$

□

Eventually we express the cross-ratio in a way which will be useful in the study of Teichmüller space. Let D_i, $i \in [|1, 4|]$ be four lines in $\mathbb{R}^2$ referred to by their slope: $z_i \in \mathbb{RP}^1$, $i \in [|1, 4|]$. In the generic case (if no z_i equals ∞) we have the following formula:

$$\boxed{r(D_1, D_2, D_3, D_4) = \frac{z_{12} z_{34}}{z_{23} z_{41}}}$$

where $z_{ij} = z_i - z_j$. The equivalence between this formula and the one with determinants is easy to prove. In the case where one of the z_i equals ∞, one has to chose a basis vector for the corresponding line and replace the z_{ij} (for all j such that z_{ij} is involved in the definition) by the determinant. With a tiny abuse of notation, we keep this definition of the cross-ratio of four points $z_i \in \mathbb{RP}^1$ in the general case because it explicitly exhibits interesting properties we will use later.

2.2 The cross ratio as a gauge theory on a graph

The cross-ratio of four points can be defined as the monodromy of a connection on a graph $\Gamma_{(0,(4))}$ corresponding to four points in $\mathbb{RP}^1$. We prove the one-to-one correspondence between the moduli space of connections on $\Gamma_{(0,(4))}$ and $\mathrm{Conf}_4^*(\mathbb{RP}^1)_+$ where the later space is the subspace of $\mathrm{Conf}_4^*(\mathbb{RP}^1)$ containing orbits of quadruples (z_1, z_2, z_3, z_4) such that (z_1, z_2, z_3) is positively oriented.

Construction of the graph

Definition 21. *A graph $\Gamma = (V, E)$ is said to be bipartite if there exists a decomposition of the set of vertices V:*

$$V = V_W \amalg V_B$$

into disjoint subsets V_W and V_B, called respectively the set of white vertices and the set of black vertices, such that every edge connects one vertex in V_W with another in V_B.

Consider the z_i on the boundary of the Poincaré disk. $\Gamma_{(0,(4))}$ is constructed in the following way:

- There is a white vertex at each of the z_i.
- There are two black vertices corresponding respectively to the triangle $\{z_1, z_2, z_3\}$ and to the triangle $\{z_1, z_3, z_4\}$.
- Let $w \in V_W$ be a white vertex and $b \in V_B$ be a black vertex. The set $\{w, b\}$ is an edge of G if and only if w is a vertex of the triangle b.

Note that the orientation of $\mathbb{RP}^1$ induces an orientation of each face of $\Gamma_{(0,(4))}$ (a formal definition of faces of fat graphs is given below while using those ideas to describe $\mathcal{T}(S)$).

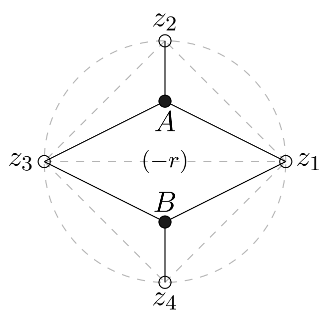


Figure 8: The unit circle corresponding to $\mathbb{RP}^1$ is dashed on the figure. The graph Γ is shown in solid lines, white and black vertices. The value $(-r)$ is assigned to the oriented face (z_1, A, z_3, B) of Γ. The two triangles are also shown in dashed lines.

Definition 22. *Let $\Gamma = (V, E)$ be a graph. An orientation on Γ is a function:*

$$s : E \to V$$

assigning to any edge its source. Choosing the source for any edge defines a target function:

$$t : E \to V$$

assigning to any edge its target.

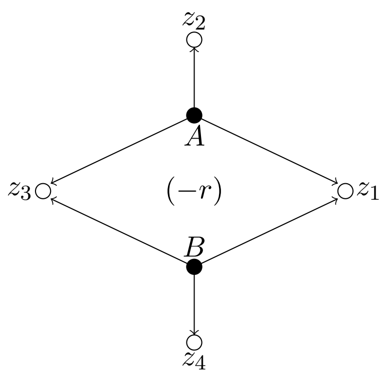


Figure 9: The oriented structure on Γ given by the bipartite structure.

In any bipartite graph there is a special orientation given by the bipartite structure:

$$s : E \to V_B$$

where V_B is the set of black vertices. We will call this orientation the *standard orientation* of the bipartite graph.

Hence $\Gamma_{(0,(4))}$ is endowed with an orientation. Let's recall some definitions and results about connections on graphs.

Principal G-connections on graphs

Definition 23. *Let $\Gamma = (V, E, s)$ be an oriented graph, and $(G, *)$ be an abelian Lie group. G is called the gauge group. A G-connexion on Γ is a function:*

$$A : E \to G$$

assigning to each oriented edge $(e \in E, s(e) \in V)$ a holonomy $A_e \in G$, such that if $e^\vee$ is the edge e with the opposite orientation ($s(e^\vee) = t(e)$):

$$A(e^\vee) = A(e)^{-1}$$

Let $\Delta(\Gamma)$ denotes the space of connections on Γ. Obviously:

$$\Delta(\Gamma) \simeq G^{\#E}$$

Definition 24. *A gauge transformation is a function:*

$$g : V \to G$$

Let $\mathfrak{G}(\Gamma)$ denotes the space of gauge transformations on Γ. One has:

$$\mathfrak{G}(\Gamma) \simeq G^{\#V}$$

Definition 25. *$\mathfrak{G}(\Gamma)$ acts on $\Delta(\Gamma)$ through:*

$$(gA)(e) = g_{t(e)} A(e) g_{s(e)}^{-1}$$

Two connections A and A' are said to be equivalent if $\exists g \in G$ such that $A' = A$ The space of equivalence classes of connections is called the moduli space of the graph G-connexions on Γ, and denoted $\mathcal{M}_G(\Gamma)$.

Theorem 2.6. *$\mathcal{M}_G(\Gamma)$ is isomorphic to the first cohomology space with coefficients in G of the graph $H^1(\Gamma, G)$.*

Proof. The set of gauge transformation is the set $\mathrm{Hom}(V, G)$ and the set of G-connections is $\mathrm{Hom}(E, G)$. We have the following exact sequence:

$$0 \leftarrow \Delta(\Gamma) \leftarrow \mathfrak{G}(\Gamma) \leftarrow 0$$

and the map $\Delta(\Gamma) \leftarrow \mathfrak{G}(\Gamma)$ is given by the action of gauge transformations on G-connections. G is abelian even if it is noted multiplicatively. Since G is a free abelian module, we have moreover that $H^1(\Gamma, G) = \mathrm{Hom}(H_1(\Gamma), G)$. □

Corollary 2.7. *Let $C_1, ..., C_n$ be n oriented loops in Γ forming a basis of the free abelian group $H_1(\Gamma, \mathbb{Z})$. The class of a connection A in $\mathcal{M}_G(\Gamma)$ is completely specified by the function:*

$$\mathrm{Mon}_A : H_1(\Gamma) \to G$$

which associates to each C_i the monodromy $\mathrm{Mon}(C_i)$ around the loop C_i in Γ.

Proof. It is sufficient to prove that Mon is well defined on $H_1(\Gamma, G)$. If C_i is the oriented loop $(e_1, ..., e_n)$:

$$\mathrm{Mon}_A(C_i) = \prod_{i=1}^{n} A_i^{\epsilon_i}$$

where $\epsilon_i = 1$ if the orientation of C_i on the i-th edge coincides with the orientation of Γ, and $\epsilon_i = 1$ otherwise. It is obvious that any gauge transformation g leaves the monodromy invariant. □

Definition 26. *Let $\mathcal{C}_G(\Gamma)$ be the category defined by:*

- $\mathrm{Ob}(\mathcal{C}_G(\Gamma))$ *is the set of G-connections on Γ*
- $\mathrm{Hom}(A_1, A_2) \neq \emptyset$ *if and only if A_1 and A_2 are equivalent. In this case* $\mathrm{Hom}(A_1, A_2)$ *is the set of gauge transformations that sends A_1 to A_2.*

Linear connexions on associate fibre bundles Let ρ be some representation of G. One can define *associate fiber bundles* to (G, ρ) on Γ.

Definition 27. *Let (W, ρ) be a representation of G. At each vertex v of Γ assign the space*

$$X_v = (G \times W)/\sim$$

where $(h, w) \sim (h', w')$ if there exists a gauge transformation g such that:

$$\begin{cases} h' = g(v)h \\ w' = \rho(g(v))^{-1}w \end{cases}$$

X is the associate fibre bundle to (G, ρ) on Γ.

Definition 28. *Let G be a graph, and W be a vector-space. A linear connection $\tilde{A}$ on $\Gamma \times W$ is a function:*

$$E(\Gamma) \to \mathrm{Gl}(W)$$

The space of linear connections on $\Gamma \times W$ is denoted $\Delta^W(\Gamma)$. A gauge transformation is a function:

$$g : V(\Gamma) \to \mathrm{Gl}(W)$$

The space of gauge transformation is denoted $\mathfrak{G}^W(\Gamma)$. $\mathfrak{G}^W(\Gamma)$ acts on $\Delta^W(\Gamma)$ by:

$$(g\tilde{A})(e) = g_{t(e)}^{-1} A(e) g_{s(e)}$$

Two linear connections are equivalent if there are in the same equivalence class under the action of $\mathfrak{G}^W(\Gamma)$.

Definition 29. *Let $\mathfrak{C}^W(\Gamma)$ be the category defined by:*

- $\mathrm{Ob}(\mathfrak{C}^W(\Gamma))$ *is the set of linear connections on $\Gamma \times W$.*
- $\mathrm{Hom}(\tilde{A}_1, \tilde{A}_2) \neq \emptyset$ *if and only if $\tilde{A}_1$ and $\tilde{A}_2$ are equivalent. In that case* $\mathrm{Hom}(\tilde{A}_1, \tilde{A}_2)$ *is the set of gauge transformations that sends $\tilde{A}_1$ to $\tilde{A}_2$.*

Provided a representation (W, ρ) of G, any G-connection A on Γ induces a linear connection $\tilde{A}$ on $\Gamma \times W$:

$$\begin{array}{rcl} \tilde{A} : E & \to & \mathrm{Gl}(W) \\ e & \longmapsto & \rho(A(e)) \end{array}$$

and a morphism of connections in $\mathcal{C}_G(\Gamma)$ induces a morphism of the corresponding linear connections. Hence:

Proposition 2.8. *Any representation (W, ρ) of G induces a contravariant functor:*

$$\mathcal{F}^\rho : \mathcal{C}_G(\Gamma) \to \mathfrak{C}^W(\Gamma)$$

If $G \sim \mathrm{Gl}(W)$ and if ρ is its standard representation, then $\mathcal{F}^\rho$ is fully faithful. $\mathcal{F}^\rho$ is essentially surjective if and only if the action of $\rho(G)$ on $\mathrm{Gl}(W)$ is transitive.

Example 2. *For $G = \mathrm{Gl}(W)$ together with its standard representation ρ_s, the functor $\mathcal{F}^{\rho_s}$ is an equivalence of categories.*

The latter definition will be our starting point for what follows. We have constructed a bipartite oriented graph Γ associated with four points in $\mathbb{RP}^1$. The goal of the remaining of this subsection is twofold:

1. First, one needs a general construction scheme of a connection on $\Gamma_{(0,(4))}$ corresponding to points $z_i \in \mathbb{RP}^1$. To each vertex will be assigned a one-dimensional vector space. To any oriented edge of the graph will be assigned a linear isomorphisms between the vector spaces corresponding to its ends. As soon as we have chosen a basis for each vector space, each morphism in matrix form becomes a real non-zero number. Hence one has constructed a connection on $\Gamma_{(0,(4))}$. The gauge freedom consists of the changes of basis in the one-dimensional vector spaces corresponding to the vertices. Since the moduli space of affine connections $\mathcal{M}^{\mathbb{R}}(\Gamma)$ is isomorphic to $\mathcal{M}_{\mathbb{R}^*}(\Gamma)$ because $\mathbb{R}^* = \mathrm{Gl}_1(\mathbb{R})$, it is parametrised by the value of the monodromy of a connection representing a class in $\mathrm{Hom}(H^1(\Gamma), \mathbb{R}^*)$. Being well defined on H^1 means that the monodromy does not depend on the gauge freedom, thus it depends only on the $\mathbb{RP}^1$, and it is an invariant of the four points in $\mathbb{RP}^1$.
2. The second objective is to show the isomorphism $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))}) \sim \mathrm{Conf}_4^*(\mathbb{RP}^1)_+$.

Construction of the connection on $\Gamma_{(0,(4))}$

- Let $v_1, ..., v_4$ label the four white vertices of $\Gamma_{(0,(4))}$ corresponding to $z_1, ..., z_4 \in \mathbb{RP}^1$. To v_i assign the line in $\mathbb{R}^2$ corresponding to z_i, and call this one-dimensional vector space D_i.
- Let v_A, v_B be the black vertices of $\Gamma_{(0,(4))}$. v_A is linked to the white vertices v_1, v_2, v_3 to which are assigned the lines D_1, D_2, D_3. The z_i are distinct hence the map:

$$D_i \oplus D_j \oplus D_k \to \mathbb{R}^2$$

corresponding to the submersion of the direct sum of the lines into $\mathbb{R}^2$ has a one-dimensional kernel, which is assigned to v_A. An element of this kernel is a triple (a_1, a_2, a_3) of vectors such that $\forall i \in [|1, 3|]$, $a_i \in D_i$, and such their sum in $\mathbb{R}^2$ is zero.

A straightforward calculus gives:

Proposition 2.9. *The equations of* $A = \ker(D_1 \oplus D_2 \oplus D_3 \to \mathbb{R}^2)$ *are given by:*

$$(x_1, x_2, x_3) \in A \Leftrightarrow x_2 = \frac{\det(x_3, x_1)}{\det(x_2, x_3)} x_1 \text{ and } x_3 = \frac{\det(x_1, x_2)}{\det(x_2, x_3)} x_1$$

$\forall i \in [|1, 4|]$, let e_i be a basis (i.e a non-zero vector) of D_i. Accordingly, if v_A is a black vertex linked to the white vertices v_i, v_j, v_k, let e_A be a non-zero vector of $\ker(D_i \oplus D_j \oplus D_k \to \mathbb{R}^2)$. For example, we may choose:

$$e_A = \sqrt{\frac{\det(e_2, e_3)}{\det(e_3, e_1)\det(e_1, e_2)}} e_1 + \sqrt{\frac{\det(e_3, e_1)}{\det(e_1, e_2)\det(e_2, e_3)}} e_2 + \sqrt{\frac{\det(e_1, e_2)}{\det(e_2, e_3)\det(e_3, e_1)}} e_3$$

and

$$e_B = \sqrt{\frac{\det(e_3, e_4)}{\det(e_1, e_3)\det(e_4, e_1)}} e_1 + \sqrt{\frac{\det(e_4, e_1)}{\det(e_3, e_4)\det(e_1, e_3)}} e_3 + \sqrt{\frac{\det(e_1, e_3)}{\det(e_3, e_4)\det(e_4, e_1)}} e_4$$

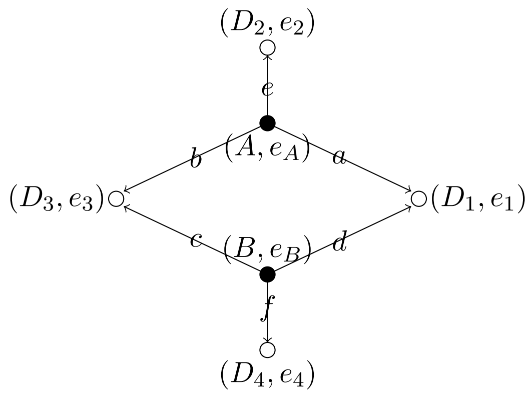


Figure 10: The linear connection on $\Gamma_{(0,(4))}$ constructed from the data of four distinct points $z_i \in \mathbb{RP}^1$.

$\forall i \in [|1, 3|]$, consider the linear map corresponding to the projection of the one-dimensional space of triples of vectors in D_1, D_2, D_3 which sum in $\mathbb{R}^2$ is zero to D_i:

$$\phi_{A,i} : A \to D_i$$

For example, for $i=1$ and for the choice of bases described above, this isomorphism is matrix form is the number:

$$\sqrt{\frac{\det(e_2,e_3)}{\det(e_3,e_1)\det(e_1,e_2)}}$$

Assign this number to the corresponding oriented edge. It forms a connexion on the graph. For example, a change of basis in A: $e_A \to e'_A = ge_A$ with $g \in \mathbb{R}^*$ modifies the connexion in the following way:

$$\left(\begin{array}{l} a \to ga \\ b \to gb \\ e \to ge \end{array} \right.$$

Conversely, if $e_1 \to e'_1 = ge_1$ then:

$$\left(\begin{array}{l} a \to \frac{a}{g} \\ d \to \frac{d}{g} \end{array} \right.$$

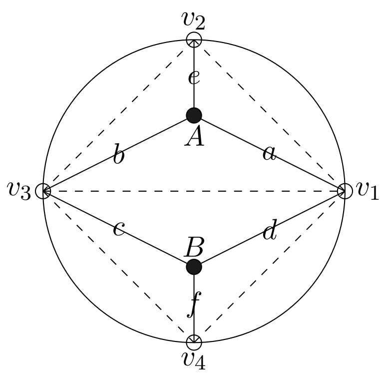
The determinant is preserved by the group $\mathrm{PSL}_2(\mathbb{R})$, hence the connection induced by the e_i is constant on the orbits of quadruple of points in $\mathbb{R}^2$ under the diagonal action of $\mathrm{PSL}_2(\mathbb{R})$.

The discussion on gauge theory on oriented graphs shows that the monodromy along the loop (v_1, v_A, v_3, v_B) only depends on the class of the connection in $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})$. Hence it does not depend on the choice of basis in any of the vector spaces assigned to the vertices of $\Gamma_{(0,(4))}$. Moreover, a simple calculus shows that:

Proposition 2.10. *The monodromy along the loop (v_1, v_A, v_3, v_B) equals the cross-ratio of the quadruple (D_1, D_2, D_3, D_4).*

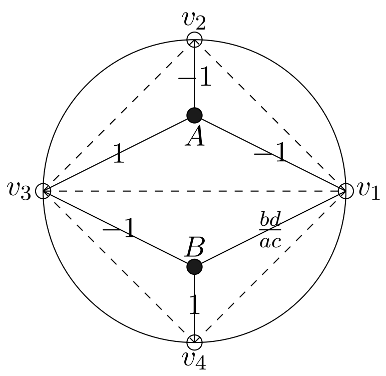
The isomorphism $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})^* \simeq \mathrm{Conf}_4^*(\mathbb{RP}^1)_+$

Proposition 2.11. *Consider the following labelling of the vertices of $\Gamma_{(0,(4))}$, and let A be an linear connection on $\Gamma_{(0,(4))}$ corresponding to the data $a,b,c,d,e,f \in \mathbb{R}^*$. Then in the equivalence class of A in $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})$, there is the connection induced by the quadruple of points $(\infty, -1, 0, \frac{-bd}{ac}) \in \mathbb{RP}^1$.*



Proof. Up to a gauge transformation with support in $\{v_A, v_B\}$ one can suppose that $abe > 0$ and $cdf > 0$. Then one can rescale the white vertices to obtain the following connection. It is exactly the one induced by the quadruple $(\infty, -1, 0, \frac{-bd}{ac})$.

□



A point in $\mathcal{M}^{\mathbb{R}}(\Gamma)$ is completely determined by the monodromy around the loop (v_1, v_A, v_3, v_B), which coincides with the cross-ratio of four points in $\mathbb{RP}^1$ for a connection corresponding to a quadruple of points (z_1, z_2, z_3, z_4) such that (z_1, z_2, z_3) and (z_1, z_3, z_4) are positively oriented. Hence we have the theorem:

Theorem 2.12. *$\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})^* \simeq \mathrm{Conf}_4^*(\mathbb{RP}^1)_+$ where $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})^*$ is the subspace of $\mathcal{M}^{\mathbb{R}}(\Gamma_{(0,(4))})$ such that the monodromy is in $\mathbb{R}\backslash\{0,1\}$.*

2.3 Application to Teichmüller spaces

Let $S_{(g,(P))}$ be a ciliated surface and let Γ be a triangulation of $S_{(g,(P))}$. The orientation on $S_{(g,(P))}$ induces a fat structure on Γ.

Definition 30. *A fat graph is a graph such that at each vertex a circular order of the edges adjacent to it is given. Any fat structure on a graph defines a set of special closed curves called the faces of the fat graph. Those closed curves are obtained by choosing an edge e and an orientation on it, and constructed in the following way. The curve starts at the source $s(e)$ of the edge e. At its target $t(e)$, the next edge is the closest one to e which is before e with respect to the fat structure (one says that the path turns left at each vertex because of the conventional choice of counterclockwise orientation at each vertex). One repeats this operation until the path comes to $s(e)$, hence obtaining a loop in the graph. The set of faces is denoted $F(\Gamma)$.*

Each face of Γ is either filled or empty according to the structure of $S_{(g,(P))}$. If $i \in F(\Gamma)$ is empty, we set $n_i = 0$ and if it is filled, $n_i = 1$. Hence the nature of the faces of Γ is encoded in the function:

$$n : F(\Gamma) \to \{0, 1\}$$

Γ is a triangulation thus any filled face of Γ is a topological triangle. A cellular decomposition of a closed surface $\tilde{S}$ is given by Γ and n, and $S_{(g,(P))}$ can be obtained as $\tilde{S}\backslash(V(\Gamma) \cup E_e(\Gamma))$ where $E_e(\Gamma)$ is the set of external edges of Γ. The set of cilia is the set of vertices which are the end of an external edge.

Definition 31. *Let $B(\Gamma)$ be the bipartite graph associated to the triangulation Γ defined by:*

- *$V_W(B(\Gamma)) = V(\Gamma)$.*
- *$V_B(B(\Gamma)) = \{i \in F(\Gamma) | n_i = 1\}$.*
- *If $v_w \in V_W(B(\Gamma))$ and $v_b \in V_B(B(\Gamma))$, $\{v_w, v_b\} \in E(B(\Gamma))$ if and only if v_w is a vertex of the triangle v_b in Γ.*

Remark 8. *The choice of a complex structure on $S_{(g,(P))}$ lifts $B(\Gamma)$ to a bipartite graph $\tilde{B}(\Gamma) \subset \mathbb{H}$. One can follow the construction of the connection on $\Gamma_{(0,(4))}$ given in the previous section to obtain a connection on $\tilde{B}(\Gamma)$. Now since this construction assigns numbers on the edges which are constant on each* $\mathrm{PSL}_2(\mathbb{R})$*-orbit of quadruples of points, this connection induces a connection on $B(\Gamma)$. Hence a complex structure on $S_{(g,(P))}$ defines a connection on $B(\Gamma)$.*

Conversely, any class of connections $\mathcal{A} \in \mathcal{M}^{\mathbb{R}}(B(\Gamma))$ is defined by the data of the monodromies along a set of loops in $B(\Gamma)$ forming a basis of $H_1(\Gamma, \mathbb{Z})$. Let $C_1, ..., C_p$ be those loops and $\forall i \in [|1, p|]$, let $m_i = -\mathrm{Mon}_i \in \mathbb{R}^*$. We have some redundancy due to the loops surrounding holes, because the monodromy around a boundary component is not related in any way to the complex structure on S. Hence, if $\sim$ is the equivalence relation on $\mathcal{M}^{\mathbb{R}}(B(\Gamma))$ given by $A \sim B$ if $\forall i \in F(\Gamma)$, $n_i = 1 \Rightarrow m_i^A = m_i^B$, let $\mathcal{M}^{\mathbb{R}}(B(\Gamma))_1$ be the quotient $\mathcal{M}^{\mathbb{R}}(B(\Gamma))/ \sim$. On each equivalence class there is a well defined function:

$$\begin{array}{rcl} n^{-1}\{1\} & \to & \mathbb{R}^* \\ i & \mapsto & m_i \end{array}$$

Thus we have the following result:

Proposition 2.13.

$$\mathcal{T}^x(S_{(g,(P))}) \simeq \bigcap_{i=1}^{p} m_i^{-1}(\mathbb{R}_+^*) \subset \mathcal{M}^{\mathbb{R}}(B(\Gamma))_1$$

which emphasise the cluster structure of the Teichmüller $\mathcal{X}$-space of $S_{(g,(P))}$: this result can be seen as the definition of a global chart on $\mathcal{T}^x(S_{(g,(P))})$ for any triangulation Γ of $S_{(g,(P))}$. The aim of the next subsection is to compute the transition functions from one of those charts to another.

2.4 Flips and permutations

Changes of a connection under a flip Since any change of triangulation can be written as a finite sequence of flips, if one knows the transition function for any flip then one is able to compute any transition function.

The modification of $B(\Gamma)$ subsequent to the flip of an edge α is obvious, however we have *a priori* no way to trivial relation between the equivalence class of a connection defined on $B(\Gamma)$ and the equivalence class of a connection defined on $B(\Gamma_\alpha)$. How can one link these two objects which are not defined on the same graph?

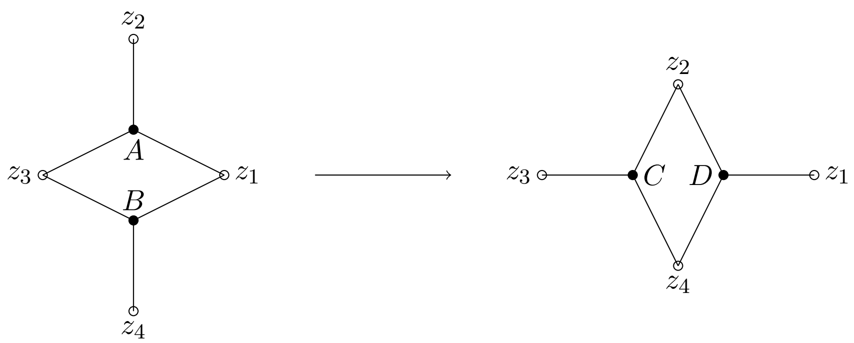


Figure 11: Variation of $B(\Gamma)$ under a flip. The complex structure on $S_{(g,(P))}$ determines the configuration of the z_i up to the action of $\mathrm{PSL}_2(\mathbb{R})$ thus the latter does not depend on the triangulation.

Lemma 2.14. *Let $z_1, z_2, z_3, z_4 \in \mathbb{RP}^1$ be four distinct points corresponding to the lines $D_1, D_2, D_3, D_4 \subset \mathbb{R}^2$. The linear map:*

$$\Phi : \bigoplus_{i=1}^{4} D_i \to \mathbb{R}^2$$

corresponds to the submersion of the direct sum of the D_i into $\mathbb{R}^2$ given by vector sum. It has a 2-dimensional kernel. For all $i \in [|1,4|]$, let $e_i \in \mathbb{R}^2$ be a non-zero vector generating D_i. A vector in $\bigoplus_{i=1}^{4} D_i$ is given by the quadruple $(x_1e_1, x_2e_2, x_3e_3, x_4e_4)$, and $\ker \Phi$ *is given by the following equations:*

$$\begin{cases} x_1 = x_2 \frac{\det(e_2,e_3)}{\det(e_3,e_1)} + x_4 \frac{\det(e_3,e_4)}{\det(e_1,e_3)} \\ x_3 = x_2 \frac{\det(e_1,e_2)}{\det(e_3,e_1)} + x_4 \frac{\det(e_4,e_1)}{\det(e_1,e_3)} \end{cases}$$

Proof. It is a straightforward calculus. □

Remark 9. *Since these equations are written in terms of determinants, they are invariant on each* $\mathrm{PSL}_2(\mathbb{R})$*-orbit of quadruples of lines in $\mathbb{R}^2$.*

Let $((x_2^A, x_4^A), (x_2^B, x_4^B))$ be a basis of $\ker \Phi$, and let x_1^Z, x_3^Z for $Z = A, B$ be the corresponding values of x_1 and x_3. In the generic case, i.e when $\forall i \in [|1,4|]$, $\forall Z = A, B$, $x_i^Z \neq 0$, this choice defines a connection on the complete bipartite graph depicted in figure 12, given by assigning to each edge of this graph the corresponding x_i^Z, and with the standard orientation of the bipartite graph.

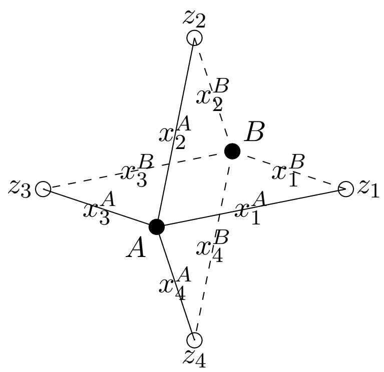


Figure 12: Each generic basis of $\ker \Phi$ defines a connection on the complete bipartite graph.

This connection can be encoded into the matrix:

$$\begin{pmatrix} x_1^A & x_2^A & x_3^A & x_4^A \\ x_1^B & x_2^B & x_3^B & x_4^B \end{pmatrix}$$

Were the basis of $\ker \Phi$ not generic, no connection can be defined on the complete bipartite graph. However, it is still possible to assign the x_i^Z to the edges of the complete bipartite graph, and this set of numbers defines a connection if restricted to the covering subgraph in which the edges are exactly the edges in the complete bipartite graph which carry a non-zero number.

Left multiplication of this matrix by a diagonal matrix corresponds to a change of basis in the vector spaces associated to the black vertices of the graph, whereas right multiplication by a diagonal matrix corresponds to a change of basis in the vector spaces associated to the white vertices of the graph.

Change of basis in $\ker\Phi$ Any matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}_2(\mathbb{R})$$

can be interpreted as a change of basis in $\ker\Phi$ if one writes:

$$M \cdot \begin{pmatrix} x_2^A & x_4^A \\ x_2^B & x_4^B \end{pmatrix} = \begin{pmatrix} x_2^C & x_4^C \\ x_2^D & x_4^D \end{pmatrix}$$

The equations of $\ker\Phi$ imply that:

$$M \cdot \begin{pmatrix} x_1^A & x_2^A & x_3^A & x_4^A \\ x_1^B & x_2^B & x_3^B & x_4^B \end{pmatrix} = \begin{pmatrix} x_1^C & x_2^C & x_3^C & x_4^C \\ x_1^D & x_2^D & x_3^D & x_4^D \end{pmatrix}$$

Example 3. *For the special choice:*

$$\begin{pmatrix} x_2^A & x_4^A \\ x_2^B & x_4^B \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & e \end{pmatrix}$$

the matrix:

$$M = \begin{pmatrix} a & c \\ f & d \end{pmatrix}^{-1}$$

sends the connection:

$$\begin{pmatrix} a & b & c & 0 \\ f & 0 & d & e \end{pmatrix}$$

which is defined on the leftmost graph of figure 11, to a connection of the form:

$$\begin{pmatrix} 1 & * & 0 & * \\ 0 & * & 1 & * \end{pmatrix}$$

which is defined on the rightmost graph of figure 11. Hence M corresponds to a flip.

Moreover, we have the following obvious proposition:

Proposition 2.15. *Since a change of basis acts linearly on the coordinates of a connection, a change of basis defines a map between equivalence classes of connections.*

Eventually the next proposition describes changes of coordinates in $\mathcal{T}^x(S)$ in terms of connections:

Proposition 2.16. *Using the equivalence between the Teichmüller space of a ciliated surface S with triangulation Γ and the subspace $\bigcap_{i=1}^p m_i^{-1}(\mathbb{R}_+^*) \subset \mathcal{M}^{\mathbb{R}}(B(\Gamma))_1$, the flip of the edge in the triangulation Γ of S corresponds to a matrix M_α in $\mathrm{PSL}_2(\mathbb{R})$. Left multiplication by M of any matrix representing a point in $\bigcap_{i=1}^p m_i^{-1}(\mathbb{R}_+^*) \subset \mathcal{M}^{\mathbb{R}}(B(\Gamma))_1$ gives a matrix which defines a point in $\bigcap_{i=1}^p m_i^{-1}(\mathbb{R}_+^*) \subset \mathcal{M}^{\mathbb{R}}(B(\Gamma_\alpha))_1$. These two equivalence classes of connections correspond to the same complex structure.*

For any choice of a basis in $\ker\Phi$, the monodromies of the connection only depend on the equivalence class of quadruples of points in $\mathbb{RP}^1$ under the action of $\mathrm{PSL}_2(\mathbb{R})$. In the generic case, since the first homology group of the complete bipartite graph is of rank 3, there are three invariants. Note that since the four points in $\mathbb{RP}^1$ are distinct, any vector in $\ker\Phi$ has at most one coordinate equal to 0. If one chooses a basis in which every vector has exactly one zero component, the first homology group of the graph on which the $\mathbb{R}^*$-connection is well defined is of rank 1 hence there is only one invariant. There are six basis of this form, corresponding to the choice of a subset of $\{z_1, z_2, z_3, z_4\}$ containing two elements. The cross-ratio of the quadruple (z_1, z_2, z_3, z_4) corresponds to the conventional choice of the subset $\{z_1, z_3\}$. Hence any other special basis can be interpreted as the cross-ratio of a quadruple which is obtained from $(z_1, z_2, z_3, z_4)$by permutation of the z_i. It is interesting to see how the cross-ratio changes under the action of $\mathfrak{S}_4$ on the quadruple (z_1, z_2, z_3, z_4).

Changes of the cross-ratio under permutations of the points There are 24 permutations of the quadruple (z_1, z_2, z_3, z_4). A priori, the cross-ratio should take 24 values as the $\mathfrak{S}_4$ acts on the quadruple of points. However, it is easy to see that the double transpositions (12)(34), (13)(24) and (14)(23) leave the cross-ratio invariant. The set of double transpositions in $\mathfrak{S}_4$ together with the identity permutation form the *Klein viergruppe* K, which is a normal subgroup of $\mathfrak{S}_4$. The quotient $\mathfrak{S}_4/K$ is isomorphic to $\mathfrak{S}_3$. Hence the cross-ratio does not take more than six values as $\mathfrak{S}_4$ acts on the quadruple of points.

There is a way of summarising this action by drawing two tetrahedron of vertices z_1, z_2, z_3, z_4. The two differs by their orientation: for example, the (123)-face of the first one is counterclockwise oriented (we say that its orientation is positive), whereas the (123)-face of the second one is clockwise oriented (and hence it has a negative orientation). We choose some edge of one of the two tetrahedra, for example the (13)-edge of the tetrahedron with positive orientation, and turn the tetrahedron in a way that this edge is at the top of the latter. Looking from above and projecting the tetrahedron on a horizontal plane, it defines a quadrilateral in the plane, with a diagonal which is the edge with which we started.

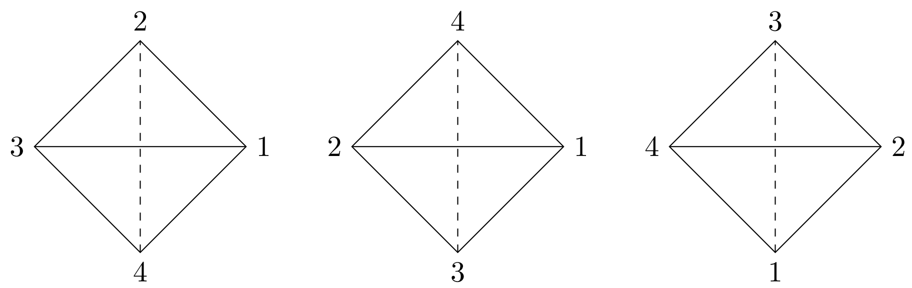


Figure 13: Quadrilaterals with triangulation obtained by projecting one of the two tetrahedra on a plane after choosing with a prescribed edge on top. The two first quadrilateral corresponds to the tetrahedron with positive orientation, the first one to the edge (13), and the second one to the edge (12), whereas the third one correspond to the choice $(2, 4)$ in the tetrahedron with negative orientation.

This manipulation allows us to assign to each edge of these tetrahedra a value of the cross-ratio of the four vertices. These values can all be expressed in terms of one, for example the one corresponding to the choice (13) in the positive tetrahedron. The values are summarised in the next figure.

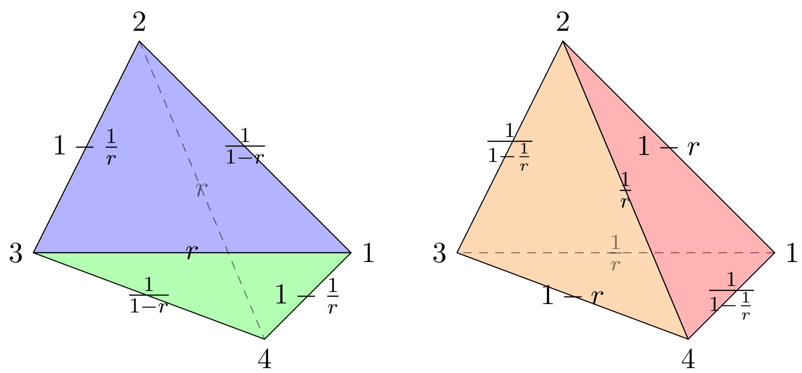


Figure 14: The two tetrahedra, with cross-ratios on the edges. The one on the left is the tetrahedron with positive orientation, the one on the right is the one with the orientation reversed.

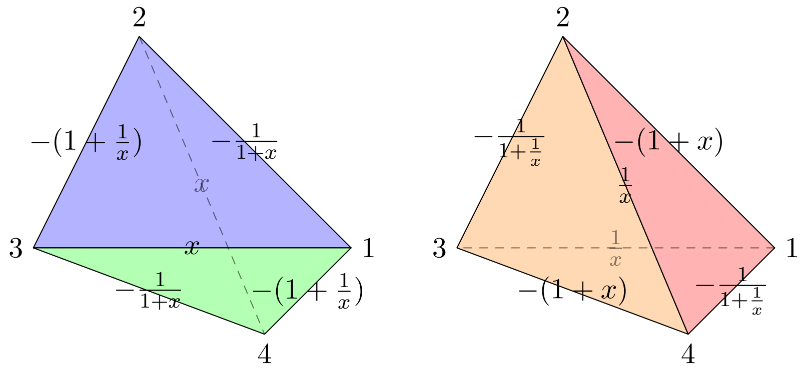


Figure 15: The two tetrahedra, with minus the cross-ratios on the edges. The one on the left is the tetrahedron with positive orientation, the one on the right is the one with the orientation reversed.

Just like before, we can construct a bipartite graph on each tetrahedron with a black vertex for each face, a white vertex for each vertex and edges as before. The monodromy of the connection induced by the choice of points in $\mathbb{RP}^1$ at each vertex only depend on one real parameter.

The tetrahedra is a good visual tool to compute a change of coordinates under a flip in the triangulation. As an example, let's study the change of coordinates induced by the following flip for the triangulation of a topological disk with five marked points.

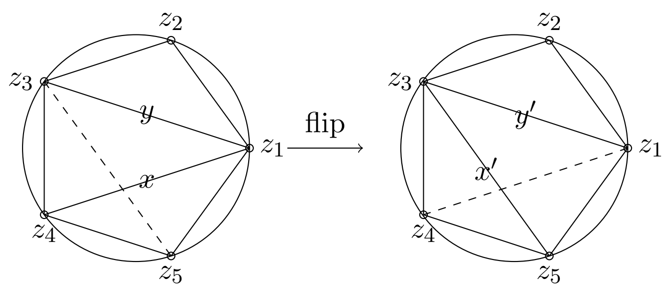


Figure 16: The two triangulations are related by the flip of an edge. The dashed edge in the tetragonal (z_1, z_3, z_4, z_5) in which the diagonal is flipped aims to represent the edge of the hidden side of the tetrahedron of vertices z_1, z_3, z_4, z_5. We directly have $x' = \frac{1}{x}$. If one thinks to the gauge theory on the bipartite graph corresponding to the tetrahedra, to compute y' we follow the path going from z_3 to z_1 through the face (z_3, z_5, z_1) then from z_1 to z_3 through (z_1, z_2, z_3). It's equivalent to go from z_3 to z_1 through the face (z_3, z_5, z_1) then from z_3 to z_1 through (z_3, z_1, z_4), then from z_1 to z_3 through (z_3, z_4, z_1) and eventually from z_3 to z_1 through (z_1, z_2, z_3). The tetrahedron model gives a factor $(1+x)$ for the first part and by definition, the factor corresponding to the second part is y. One has to be careful with the multiplication by (-1) if one computes minus the cross-ratio, but one eventually obtains $y' = y(1 + x)$. This construction can be generalised to the Teichmüller space of any ciliated hyperbolic surface.

3 Grassmann algebras, supermanifolds and the group $\mathrm{SpO}(2|1)(\mathbb{R})$

3.1 Superalgebras, supermodules and Grassmann algebras

In this subsection and the following one, we will follow [Man88].

The supercommutator

Definition 32. *Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded ring, in which the element 2 is invertible. Let $a \in A$ be a homogeneous element of degree ϵ. We note $\tilde{a} = \epsilon$. If $\tilde{a} = 0$, a is said to be even, and if $\tilde{a} = 1$, it is said to be odd. The supercommutator of two homogeneous elements $a, b \in A$ is defined by:*

$$[a, b] = ab - (-1)^{\tilde{a}\tilde{b}} ba$$

and this definition extends to non-homogeneous elements by additivity.

Remark 10. *The additional sign that appears in the definition of the supercommutator in not relevant in characteristic 2, hence the assumption about multiplication by 2 in A. In modules, we will suppose as well that multiplication by 2 is an isomorphism.*

Definition 33. *We say that $a, b \in A$ supercommute if:*

$$[a, b] = 0$$

A is supercommutative if any two of its elements supercommute. It implies in particular that any odd element is nilpotent since it anticommutes with itself.

Remark 11. *If A is associative then the supercommutator satisfies the following identity:*

$$[a, [b, c]] + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}[b, [c, a]] + (-1)^{\tilde{c}(\tilde{a}+\tilde{b})}[c, [a, b]] = 0$$

which is called the super-Jacobi identity.

As an example let's study Grassmann algebras. They are supercommutative rings which are the cornerstone of the construction of smooth supermanifolds.

Grassmann algebras

Definition 34. *The Grassmann algebra with $L \in \mathbb{N} \cup \{\infty\}$ generators over $\mathbb{K}$ is defined by:*

$$G_L(\mathbb{K}) = < 1, \theta_1, ..., \theta_L | \forall i \in [|1; L|]\ 1 \cdot \theta_i = \theta_i \cdot 1, \text{ and } \forall i, j \in [|1; L|]\ \theta_i \cdot \theta_j = -\theta_j \cdot \theta_i >$$

For example, $G_\infty(\mathbb{K})$ is the algebra over $\mathbb{K}$ generated by $1, \theta_1, \theta_2, ...$ and subject to the relations:

$$\forall i \in \mathbb{N}^*\ 1 \cdot \theta_i = \theta_i \cdot 1$$

$$\forall i, j \in \mathbb{N}^*\ \theta_i \cdot \theta_j = -\theta_j \cdot \theta_i$$

In the sequel we will shorten $G_\infty(\mathbb{K})$ to $G(\mathbb{K})$.

Example 4. *Let $L \in \mathbb{N}$. The Grassmann algebra $G_L(\mathbb{R})$ is isomorphic (as an algebra) to the exterior algebra $\bigwedge V$ of an L-dimensional real vector space V.*

Let $L \in \mathbb{N} \cup \{\infty\}$, and let I be the following set:

$$I = \{\emptyset\} \cup \bigcup_{k \in \mathbb{N}^*} I^k$$

where:

$$I^k = \{(i_1, ..., i_k) | i_1 < ... < i_k \in [|1; L|]\}$$

is the set of multi-indices of length k. By convention, we write: $I^0 \equiv \{\emptyset\}$. If $(i_1, ..., i_k) \in I^k$, we set:

$$\theta_{(i_1,...,i_k)} \equiv \theta_{i_1} \theta_{i_2} ... \theta_{i_k}$$

$x \in G_L(\mathbb{K})$ can be uniquely written as:

$$x = \sum_{\Gamma \in I} x_\Gamma \theta_\Gamma$$

Let $I_x \subset I$ be the finite set of multi-indices Γ for which x_Γ is nonzero.

Definition 35. *Writing:*

$$x = x^\# + \sum_{\Gamma \in I - \{\emptyset\}} x_\Gamma \theta_\Gamma$$

one says that $x^\#$ is the body of x, whereas

$$s(x) = x - x^\# = \sum_{\Gamma \in I - \{\emptyset\}} x_\Gamma \theta_\Gamma$$

is its soul.

Remark 12. *If $L \leq L' \in \mathbb{N} \cup \{\infty\}$, we have a canonical injection $G_L\mathbb{R} \to G_{L'}\mathbb{R}$. Hence $G(\mathbb{K})$ is the inductive limit of the family $(G_L(\mathbb{K}))_{L \in \mathbb{N}}$, as the following diagram commutes:*

$$\begin{array}{ccccccccc} ... & \longrightarrow & G_{L-1}(\mathbb{K}) & \longrightarrow & G_L(\mathbb{K}) & \longrightarrow & G_{L+1}(\mathbb{K}) & \longrightarrow & ... \\ & & & \searrow & \downarrow & \swarrow & & & \\ & & & & G(\mathbb{K}) & & & & \end{array}$$

Definition 36. *The Grassmann algebra with $L \in \mathbb{N} \cup \{\infty\}$ generators is $\mathbb{Z}/2\mathbb{Z}$-graded:*

$$G_L(\mathbb{K}) = G_L(\mathbb{K})_0 \oplus G_L(\mathbb{K})_1$$

Elements of $G_L(\mathbb{K})_0$ are linear combinations of products of an even number of θs, and are said to be even, whereas those of $G_L(\mathbb{K})_1$ are linear combinations of products of an odd number of θs and are said to be odd. An element $x \in G_L(\mathbb{K})$ is the sum of its even part and its odd part: $x = x_0 + x_1$. The grading:

$$|\cdot| : G_L(\mathbb{K})_0 \oplus G_L(\mathbb{K})_1 \to \mathbb{Z}/2\mathbb{Z}$$

sends any even element to zero and any odd one to one. The Grassmann algebra is super-commutative since for homogeneous elements a and b:

$$ab = (-1)^{|a||b|} ba$$

It will be useful to decompose I into its even and odd pieces: $I = I_0 \cup I_1$, with $I_0 = \bigcup_{k \in \mathbb{N}} I^{2k}$ and $I_1 = \bigcup_{k \in \mathbb{N}} I^{2k+1}$. We write $I_x = I_{x,0} \cup I_{x,1}$ as well.

The following theorem is very useful for the construction of $\mathcal{T}_{\mathrm{SpO}(2|1)(\mathbb{R})}(S)$:

Theorem 3.1. *Let $L \in \mathbb{N} \cup \{\infty\}$ and $x \in G_L(\mathbb{K})$*

1. *If the body of x is nonzero, x admits a two-sided inverse in $G_L(\mathbb{K})$.*
2. *If the body of x is strictly positive, x admits a square root y which is uniquely determined by requiring that the body of the latter is also strictly positive.*

We prove the first part through two elementary lemmas:

Lemma 3.2. *Let $L \in \mathbb{N}^*$, and $x \in G_L(\mathbb{K})$. If y is right-inverse of x, and if $x_1 y_1 = 0$ then y is also a left-inverse of x.*

Proof. Let $x, y \in G_L(\mathbb{K})$ such that $xy = 1$. Writing $x = x_0 + x_1$ and $y = y_0 + y_1$, the product $xy = 1$ is rewritten as:

$$x_0 y_0 + x_0 y_1 + x_1 y_0 = 1$$

Using super-commutativity:

$$xy = y_0 x_0 + y_1 x_0 + y_0 x_1 = yx = 1$$

□

Lemma 3.3. *Let $L \in \mathbb{N}^*$, $x', x'' \in G_L(\mathbb{K})$ and $x = x' + x''\theta_{L+1} \in G_{L+1}(\mathbb{K})$. If x' is invertible in $G_L(\mathbb{K})$ with inverse $(x')^{-1}$ and if $(x')_1((x')^{-1})_1 = 0$ then x is invertible in $G_{L+1}(\mathbb{K})$ with inverse given by:*

$$x^{-1} = (x')^{-1}(1 + x''[((x')^{-1})_1 - ((x')^{-1})_0]\theta_{L+1})$$

Moreover:

$$x_1(x^{-1})_1 = 0$$

Proof. The first part of the lemma is a simple computation. For the second part, note that:

$$x'_1(x'^{-1})_0 = -x'_0(x'^{-1})_1$$

which gives the result if one recalls that super-commutativity implies anti-commutativity of the odd numbers θ_{L+1} and $(x'^{-1})_1$. □

End of the proof of 3.1.1. We conclude by recurrence oven L. If $x = a + b\theta$ with $a, b \in \mathbb{K}^*$, x^{-1} is given by:

$$x^{-1} = \frac{1}{a} - \frac{b}{a^2}\theta$$

Now if $x \in G_L(\mathbb{K})$ with nonzero body, writing $x = x' + x''\theta_L$ with $x', x'' \in G_{L-1}(\mathbb{K})$, we see that x' has a nonzero body as well, thus an inverse, and the latter lemma concludes the proof. □

The proof of the second part of theorem 3.1 is easily done by recurrence as well.

Modules Let A be any $\mathbb{Z}_2$-graded ring. The axioms for left A-modules, right A-modules and A-bimodules are the $\mathbb{Z}_2$-graded version of the usual axioms.

Definition 37. *Suppose that A is supercommutative, and that S is endowed with a structure of right A-module and left A-module. The two structures are said to be compatible if:*

$$as = (-1)^{\tilde{a}\tilde{s}} sa$$

Hence on a supercommutative ring there is an equivalence between left A-modules, right A-modules and A-bimodules. If $S = S_0 \oplus S_1$, then S_0 and S_1 are A_0-submodules of S.

Definition 38. *An additive mapping of A-modules:*

$$f : S \to T$$

is said to be an even homomorphism if it preserves the grading and is A-linear. Then $f(as) = af(s)$ implies $f(sb) = f(s)b$ and conversely. Hence if A is supercommutative an even morphism of left A-modules or right A-modules is automatically a morphism of A-bimodules. Let $\mathrm{Hom}_0(S,T)$ be the additive group of even morphisms from S to T.

Definition 39. *An additive mapping of A-modules:*

$$f : S \to T$$

is an odd homomorphism if it reverses the grading and if:

$$f(as) = (-1)^{\tilde{a}} af(s)$$

and:

$$f(sb) = f(s)b$$

Again if A is supercommutative an odd morphism of left A-modules or right A-modules is automatically a morphism of A-bimodules. Let $\mathrm{Hom}_1(S,T)$ be the additive group of even morphisms from S to T.

We set:

$$\mathrm{Hom}(S,T) = \mathrm{Hom}_0(S,T) \oplus \mathrm{Hom}_1(S,T)$$

Parity change

Definition 40. *Let S be a left-or-right A-module. We define the module ΠS by setting $(\Pi S)_0 = S_1$ and $(\Pi S)_1 = S_0$. Addition in ΠS is the same as in S, right-multiplication by A is the same as in S, and left-multiplication is given by:*

$$a(\Pi s) = (-1)^{\tilde{a}} \Pi(as)$$

where $\Pi s \in \Pi S$ is the element corresponding to $s \in S$.

If A is supercommutative, Π translates consistent structure on S to consistent structures on ΠS. Let $f : S \to T$ be a morphisms of A-modules (of the same category), and let $f^{\Pi} : \Pi S \to \Pi T$ denote the morphism which coincides with f as a mapping of sets. This makes the association $S \to \Pi S$ a functor, for which $\Pi^2 = id$.

Free A-modules If A is a $\mathbb{Z}_2$-graded ring, A with left, right, or two-sided multiplication is an A left, right or bi-module.

Definition 41. *A free A-module of rank $p|q$ is an A-module isomorphic to:*

$$A^{p|q} := A^p \oplus (\Pi A)^q$$

It has a free system of generators, p of which are even and q of which are odd.

As in the commutative case, if A is supercommutative, then two free modules of different ranks are not isomorphic (as one can see be taking the quotient of the ring by one of its maximal ideals).

Matrices in linear superalgebra Morphisms between free A-modules can be given by matrices. Matrices act on the left on columns and on the right on rows. Let's give the formal definition of a matrix format.

Definition 42. *A matrix format is the data of two finite sets which are divided into non-intersecting subspaces of even and odd elements: $I = I_0 \cup I_1$ and $J = J_0 \cup J_1$. A matrix in the given format with values in the ring A is a map:*

$$a : I \times J \to A$$

The parity of the position (ij) in the given matrix format is $\tilde{i} + \tilde{j}$, where $\tilde{i} = e if i \in I_e$. A matrix is even if for all i, j:

$$\tilde{a_{ij}} = \tilde{i} + \tilde{j}$$

A matrix is odd if for all i, j:

$$\tilde{a_{ij}} = \tilde{i} + \tilde{j} + 1$$

Definition 43. *The matrix format is said standard if $I_0 = \{1, ..., m\}$, $I_1 = \{m + 1, ..., m + n\}$, $J_0 = \{1, ..., p\}$ and $J_1 = \{p + 1, ..., p + q\}$. The set of matrices in standard format with elements from the ring A is denoted by $\mathcal{M}(m|n, p|q; A)$. It has the structure of a $\mathbb{Z}_2$-graded A-module. Together with the usual matrix multiplication, it is naturally isomorphic to* $\mathrm{Hom}(A^{p|q}, A^{m|n})$. *Composition of morphisms corresponds to matrix multiplication:*

$$\mathcal{M}(m|n, p|q; A) \times \mathcal{M}(p|q, r|s; A) \to \mathcal{M}(m|n, r|s; A)$$

3.2 The algebro-geometric approach to supermanifolds

Definition 44. *A superspace is a pair $(M, \mathcal{O}_M)$ where M is a topological space and $\mathcal{O}_M$ a sheaf of supercommutative rings such that the stalk $\mathcal{O}_x = \mathcal{O}_{M,x}$ at any point $x \in M$ is a local ring.*

Definition 45. *A superspace $(M, \mathcal{O}_M)$ such that $(\mathcal{O}_M)_0 = \mathcal{O}_M$ and $(\mathcal{O}_M)_1 = \{0\}$ is said to be purely even.*

Example 5. *Any differential or analytic manifold, analytic spaces or schemes are purely even superspaces.*

Definition 46. *A morphism of superspaces $f : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a pair (ϕ, ψ) where:*

$$\phi : M \to N$$

is a continuous mapping of topological spaces, and:

$$\psi : \mathcal{O}_N \to \phi_*(\mathcal{O}_M)$$

is a morphism of sheaves of rings such that for any $x \in M$:

$$\psi_x(\mathfrak{m}_{\phi(x)}) \subset \mathfrak{m}_x$$

Grading and reduced superspace Let $(M, \mathcal{O}_M)$ be a superspace.

Remark 13. *$(M, (\mathcal{O}_M)_0)$ is a purely even superspace, and the map $(\mathrm{id}_M, (\mathcal{O}_M)_0 \hookrightarrow \mathcal{O}_M)$ is the projection of $(M, \mathcal{O}_M)$ onto its "even quotient".*

Definition 47. *Let J_M be the sheaf of ideal defined by:*

$$J_M = (\mathcal{O}_M)_1 + (\mathcal{O}_M)_1^2$$

We set:

$$\mathrm{Gr}_i \mathcal{O}_M = J_M^i / J_M^{i+1}$$

For $i = 0$, $\mathrm{Gr}_0 \mathcal{O}_M = \mathcal{O}_M / J_M$ is a sheaf of rings, and $M_r := (M, \mathrm{Gr}_0 \mathcal{O}_M)$ is a purely even superspace. The map $(\mathrm{id}_M, \mathcal{O}_M \to \mathcal{O}_M / J_M)$ is a closed embedding $M_r \to M$.

Definition 48.
- *A superspace $(M, \mathcal{O}_M)$ is said to be decomposable over M^0 if it is isomorphic to $(M^0, S_{\mathcal{O}_{M^0}}(\mathcal{E}))$ where $(M^0, \mathcal{O}_{M^0})$ is a purely even superspace and $\mathcal{E}$ is a locally free sheaf of $\mathcal{O}_{M^0}$-modules of purely odd rank $0|q$, and where $S_{\mathcal{O}_{M^0}}(\mathcal{E})$ is the symmetric algebra of $\mathcal{E}$ over $\mathcal{O}_{M^0}$, that is, the algebra defined as the quotient of the tensor algebra of the module $\mathcal{E}$ modulo the ideal generated by the supercommutators $t_1 \otimes t_2 - (-1)^{\tilde{t_1}\tilde{t_2}} t_2 \otimes t_1$.*
- *A superspace $(M, \mathcal{O}_M)$ is said to be decomposable if it is decomposable over M_r.*
- *A superspace $(M, \mathcal{O}_M)$ is said to be locally decomposable if for every point $x \in M$ there is an open neighbourhood U of x such that $(U, (\mathcal{O}_M)_{|U}$ is decomposable.*

Definition 49. *A differentiable, analytic or algebraic supermanifold is defined to be a locally decomposable superspace $(M, \mathcal{O}_M)$ such that M_r is isomorphic to an ordinary (purely even) manifold of the appropriate class.*

Let $(M, \mathcal{O}_M)$ be a supermanifold. Let $x \in M$, and let (ξ^i) be a local basis of the sections of the sheaf realising the local decomposition. Locally, any function on M (i.e a local section of the structure sheaf) can be represented in the form:

$$f = \sum_\alpha f_\alpha(x) \xi^\alpha$$

where the $f_\alpha(x)$ are local functions on M_r and where $\xi^\alpha = (\xi^1)^{\alpha_1} ... (\xi^n)^{\alpha_n}$. The value of f at $x \in M$ is defined to be f mod $\mathfrak{m}_x$. An (irreducible) supermanifold has an invariantly defined superdimension $m|n$. Hence supermanifolds can be represented as superspaces with local systems of local independent coordinates $(x^1, ..., x^m, \xi^1, ..., \xi^n)$.

Definition 50. *Let $U \subset \mathbb{R}^m$ a connected domain with coordinates functions $(x^1, ..., x^m)$, and with $(\xi^1, ..., \xi^n)$ a basis of sections of the trivial sheaf of rank $0|n$ on this domain. We assume that $\mathcal{O}_U$ is the sheaf of $\mathcal{C}^\infty$-functions. The superspace:*

$$U^{m|n} = (U, \mathcal{O}_U[\xi^1, ..., \xi^m])$$

is called a differentiable superdomain with coordinate system $(x^1, ..., x^m, \xi^1, ..., \xi^m)$ on it.

To conclude this section we state the proposition 8 of the first paragraph of chapter 4 of [Man88]. It ensures that the pasting morphisms can be written in terms of local coordinates, just as in the ordinary geometric categories.

Proposition 3.4. *Let $U^{m|n}$ and $V^{m|n}$ two superdomains of the same type, with respective coordinate systems (x, ξ) and (y, η). There are natural bijections between the following sets:*

- $\mathrm{Hom}(U,V)$ *in the category of superspaces*
- $\mathrm{Hom}(\mathcal{O}(U),\mathcal{O}(V))$ *in the category of commutative superalgebras (over* $\mathbb{R}$ *or* $\mathbb{C}$*).*
- *The subset of* $\mathcal{O}_0(U)^m \oplus \mathcal{O}_1(U)^n$ *consisting of the tuples of functions* $(y^i(x,\xi),\eta^j(x,\xi))$ *such that for* $x \in U$ *we have* $y(x,0) \in V$*.*

3.3 The concrete approach to supermanifolds

In this subsection we mainly follow [Rog07].

The concrete approach to real supermanifolds is based on the peculiar superspace $\mathbb{R}^{m|n}$ defined as the $G(\mathbb{R})_0$-module:

$$\mathbb{R}^{m|n} = (G(\mathbb{R})_0)^{\times m} \times (G(\mathbb{R})_1)^{\times n}$$

The map:

$$\#^{m|n} : \mathbb{R}^{m|n} \to \mathbb{R}^m$$

which associates to each tuple $(x^1,...,x^m,\xi^1,...,\xi^n)$ in $\mathbb{R}^{m|n}$ the tuple $((x^1)^\#,...,(x^m)^\#)$ in $\mathbb{R}^m$ is called the reduction map. Accordingly, the latter tuple is called of reduction of the former one.

The DeWitt topology $\mathbb{R}^{m|n}$ can be endowed with a non-Hausdorff topology called the *DeWitt topology.*

Definition 51. *A subset* $U \subset \mathbb{R}^{m|n}$ *is open for the DeWitt topology if and only if* U *is the pre-image of an open set in* $\mathbb{R}^m$ *be the map* $\#$*.*

G^∞ and H^∞ functions on $\mathbb{R}^{m|n}$

Definition 52. *Let* $L \in \mathbb{N}$*. The projection:*

$$p_L : G(\mathbb{R}) \to G_L(\mathbb{R})$$

is defined as the $\mathbb{R}$*-linear map which is the identity on every generator* θ_i *of* $G(\mathbb{R})$ *for* $i \leq L$ *and which sends any generator* θ_i *to* 0 *if* $i > L$*.*

If $U \subset \mathbb{R}^m$, and if $f : U \to G(\mathbb{R})$, f is said to be $\mathcal{C}^\infty$ if $\forall L \in \mathbb{N}$, $p_L \circ f : U \to G_L(\mathbb{R})$ is $\mathcal{C}^\infty$. Let's denote by $\mathcal{C}^\infty(U, G(\mathbb{R})$ this set of $\mathcal{C}^\infty$ functions.

Definition 53. *Let* $f \in \mathcal{C}^\infty(U, G(\mathbb{R})$*. We define the function* $\hat{f} : (\#^{m|0})^{-1}(U) \to G(\mathbb{R}$ *defined by:*

$$\hat{f}(x) := \sum_{i_1,...,i_m=0}^{\infty} \frac{1}{i_1!...i_m!}\partial_1^{i_1}...\partial_m^{i_m} f(\#^{m|n}(x))s(x^1)^{i_1}...s(x^m)^{i_m}$$

Now we can define G^∞ and H^∞ functions on $\mathbb{R}^{m|n}$.

Definition 54. *Let* U *be an open set of* $\mathbb{R}^{m|n}$ *for the DeWitt topology.* $f : U \to G(\mathbb{R})$ *is* G^∞ *on* U *if and only if there exists a collection* $\{f_\alpha\}$ *of functions in* $\mathcal{C}^\infty(\#^{m|n}(U), G(\mathbb{R}))$ *such that:*

$$f(x,\xi) = \sum \hat{f}_\alpha(x)\xi^\alpha$$

This expression is the Grassmann analytic expansion of f *and the* f_α *are the Grassmann analytic coefficients of* f*.*

Definition 55. *Let U be an open set of $\mathbb{R}^{m|n}$ for the DeWitt topology. $f : U \to G(\mathbb{R})$ is H^∞ on U if and only if there exists a collection $\{f_\alpha\}$ of functions in $\mathcal{C}^\infty(\#^{m|n}(U), \mathbb{R})$ such that:*

$$f(x,\xi) = \sum \hat{f}_\alpha(x)\xi^\alpha$$

This expression is the Grassmann analytic expansion of f and the f_α are the Grassmann analytic coefficients of f.

The latter definition is more restrictive, since any H^∞ function is G^∞, but the controverse is not true.

DeWitt supermanifolds Let K be any class of supersmooth functions, i.e G^∞ or H^∞.

Definition 56. *Let M be a set, and $m, n \in \mathbb{N}$.*

1. *An $((m|n), K)$-chart on M is a pair (V, ϕ), where $V \subset M$, and ϕ is a bijection from V to U any open subset of $\mathbb{R}^{m|n}$.*
2. *An $((m|n), K)$-atlas on M is a collection of charts $\{(V_i, \phi_i), i \in I\}$ such that $\cup_{i\in I} V_i\} = M$ and for $i, j \in I$, $\phi_i(V_i \cap V_j)$ and $\phi_j(V_i \cap V_j)$ are open, and the map $\phi_i \circ \phi_j^{-1}$ is of class K.*
3. *An $((m|n), K)$ DeWitt super-premanifold is a set M together with a maximal $((m|n), K)$-atlas.*

The super-premanifold structure on M endows it with a topology, which is again non-Hausdorff. The relation on M given by $(p \sim q)$ if and only if there exists a chart (V, ϕ) such that $p, q \in V$ and have the same reduction is an equivalence relation. DeWitt and Bachelor (see [Bou13]) have proven the following theorem:

Theorem 3.5. *The space $M^\# = M/\sim$ is locally diffeomorphic to $\mathbb{R}^n$. It is called the body of M, and the canonical projection from M to $M^\#$ is denoted by $\#$.*

Definition 57. *A DeWitt supermanifold is a DeWitt super-premanifold whose body is a classical manifold.*

To conclude, let's state the theorem of Alice Rogers which is the proposition 8.4.1 in [Rog07] linking the two different approaches to supermanifolds.

Theorem 3.6. *Let M be an H^∞ supermanifold of dimension $(m|n)$, and let A be the sheaf of super algebras on $M^\#$ given by:*

$$A(V) = H^\infty(\#^{-1}(V))$$

Then $(M^\#, A)$ is a supermanifold in the sense of the previous subsection.

3.4 Bilinear forms

Let A be a supercommutative ring and T_1, T_2 be A-modules.

Definition 58. *A bilinear form is a map:*

$$b : T_1 \times T_2 \to A$$

which is biadditive and homogeneous, and such that:

$$\begin{cases} b(at_1, t_2) = (-1)^{\tilde{a}\tilde{b}} a b(t_1, t_2) \\ b(t_1, t_2 a) = b(t_1, t_2) a \end{cases}$$

A bilinear form b defines a morphism:

$$\begin{array}{ccc} T_1 & \to & T_2^* \\ t_1 & \mapsto & b(t_1, \cdot) \end{array}$$

b is said to be non-degenerate if this map is an isomorphism.

Definition 59. *Let $b : T_1 \times T_2 \to A$ be a bilinear form. The map:*

$$b^\tau : T_1 \times T_2 \to A$$

given by:

$$b^\tau(t_1, t_2) = (-1)^{\tilde{t_1}\tilde{t_2}} b(t_2, t_1)$$

is a bilinear form of the same parity as b. We have $(b^\tau)^\tau = b$.

Definition 60. *Let A be a supercommutative ring and T be an A-module. A bilinear form $b : T \times T \to A$ is said to symmetric if $b^\tau = b$ and antisymmetric if $b^\tau = -b$*

Definition 61. *If T_1 and T_2 are free A-modules with respective bases (e_i) and (e'_j), any bilinear form $b = T_1 \times T_2 \to A$ can be determined by its Gram matrix which is the matrix with entries:*

$$b_{ij} = (-1)^{\tilde{b}\tilde{e_i}} b(e_i, e'_j)$$

Remark 14. *Note that if $b : T_1 \times T_2 \to A$ is a bilinear form with Gram matrix B written in standard form:*

$$B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$$

the Gram matrix B^τ of the form b^τ, with respect to the same bases, has the form:

$$B^\tau = \begin{pmatrix} B_1^t & B_3^t \\ B_2^t & -B_4^t \end{pmatrix}$$

if $\tilde{B} = 0$, and:

$$B^\tau = \begin{pmatrix} B_1^t & -B_3^t \\ -B_2^t & -B_4^t \end{pmatrix}$$

if $\tilde{B} = 1$.

Definition 62. *Let A be a supercommutative ring and T be a free A-module. A bilinear form $b : T \times T \to A$ is said to be of type* SpO *if its Gram matrix in standard form satisfies:*

$$B = \begin{pmatrix} B_1 = -B_1^t & B_2 \\ B_3 = -B_2^t & B_4 = B_4^t \end{pmatrix}$$

3.5 Supertranspose and Berezinian

Supertranspose

Definition 63. *Let M be a matrix in standard form:*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

describing an even or odd morphism $b : S \to T$ of free A-modules. There exists a natural morphism $b^* : T^* \to S^*$ such that:

$$(b^*(t^*), s) = (-1)^{\tilde{b}\tilde{t}}(t^*, b(s))$$

for all $t^* \in T^*$ and $s \in S$. If M is even then:

$$M^{\mathrm{st}} = \begin{pmatrix} A^t & C^t \\ -B^t & D^t \end{pmatrix}$$

and if M is odd:

$$M^{\mathrm{st}} = \begin{pmatrix} A^t & -C^t \\ B^t & D^t \end{pmatrix}$$

Proposition 3.7.

- $(B + C)^{\mathrm{st}} = B^{\mathrm{st}} + C^{\mathrm{st}}$
- $(BC)^{\mathrm{st}} = (-1)^{\tilde{B}\tilde{C}} C^{\mathrm{st}} B^{\mathrm{st}}$
- $(\mathrm{st})^2 \neq \mathrm{id}$
- $(\mathrm{st})^4 = \mathrm{id}$

Berezinian

Definition 64. *Let A be a supercommutative ring and let $\mathrm{Gl}(p|q, A)$ be the group of invertible even automorphisms of $A^{p|q}$. One can define a map:*

$$\mathrm{Ber} : \mathrm{Gl}(p|q, A) \to A_0$$

given by the following formula. If $M \in \mathrm{Gl}(p|q, A)$ is written in standard form:

$$Ber(M) = \det\big(M_1 - M_2 M_4^{-1} M_3\big) \det\big(M_4^{-1}\big)$$

Let's state theorem 3.3.5 of [Man88]:

Theorem 3.8. *The map $\mathrm{Ber} : \mathrm{Gl}(p|q, A) \to \mathrm{Gl}(1|0, A_0)$ is a group homomorphism which agrees with the determinant when $q = 0$.*

Moreover the Berezinian has the following properties:

Proposition 3.9.

- *If S is a free A-module of finite rank, and $b \in \mathrm{Gl}(S)$ then we set $Ber(b)$ to be the Berezinian of any matrix in standard form representing b. $\mathrm{Ber}(b)$ does not depend on the choice of basis for S.*
- $\mathrm{Ber}(B^{\mathrm{st}}) = \mathrm{Ber}(B)$
- $\mathrm{Ber}(b \oplus c) = \mathrm{Ber}(B)\mathrm{Ber}(C)$

3.6 $\mathrm{SpO}(2|1)(\mathbb{R})$

Definition 65. *Let b be the bilinear form of type SpO defined on $G(\mathbb{R})^{2|1}$ through its Gram matrix:*

$$B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

The group $\mathrm{SpO}(2|1)(\mathbb{R})$ is the subgroup of $\mathrm{Gl}(2|1, G(\mathbb{R}))$ of even matrices preserving b:

$$B^{st} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Proposition 3.10. *Equivalently, the group* $\mathrm{SpO}(2|1)(\mathbb{R})$ *is the group of* 3×3 *matrices of the form:*

$$B = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$$

which satisfies:

$$\begin{cases} a,b,c,d,e \in G(\mathbb{R})_0 \\ \alpha,\beta,\gamma,\delta \in G(\mathbb{R})_1 \\ ad-bc-\alpha\beta = e^2+2\gamma\delta = 1 \\ a\delta - c\gamma - e\alpha = b\delta - d\gamma - e\beta = 0 \\ \mathrm{Ber}(B) = 1 \end{cases}$$

Remark 15. *For* $e_1, e_2 \in \mathbb{R}^{2|1} := (G(\mathbb{R})_0)^{2|1}$ *we will write:*

$$< e_1, e_2 >= b(e_1, e_2)$$

3.7 $\mathbb{H}^s$ and $\mathbb{P}^{1|1}$

Real super projective spaces Let $(\mathbb{R}^{n+1|m})^* = \#^{-1}(\mathbb{R}^{n+1}\backslash\{0\})$. The relation $\sim$ defined by:

$$(x|\zeta) \sim (x'|\zeta')$$

if $\exists l \in G(\mathbb{R})_0^\times$ such that:

$$(x|\zeta) = (l\cdot x'|l\cdot\zeta')$$

is an equivalence relation. The quotient space $(\mathbb{R}^{n+1|m})^*/\sim$ is called the real super projective n-space with m odd coordinates and is denoted by $\mathbb{P}^{n|m}$. It can be endowed with a structure of deWitt or algebraic supermanifold through the straightforward generalisation of projective charts in the classical case.

Definition 66. *Let* $\mathbb{C}^{1|1} = G(\mathbb{C})_0 \times G(\mathbb{C})_1$ *be the supermanifold with structure sheaf the superholomorphic functions (see [Bou13]).* $\mathbb{H}^s$ *is the subset of* $\mathbb{C}^{1|1}$ *such that* $(z|\zeta) \in \mathbb{H}^s$ *if* $\mathrm{Im}(z^\#) > 0$.

The boundary of $\mathbb{H}^s$ is defined to be $\mathbb{P}^{1|1}$. The latter is covered by two charts. We will use the following notation. Let:

$$\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \in (\mathbb{R}^{2|1})^*$$

The corresponding element in $\mathbb{P}^{1|1}$ is denoted by:

$$\begin{pmatrix} \frac{x}{y} \\ \frac{\theta}{y} \end{pmatrix} \in \mathbb{R}^{2|1}$$

In the special case where $y = 0$, we note:

$$\begin{pmatrix} \infty \\ \frac{\theta}{x} \end{pmatrix} \in \mathbb{R}^{2|1}$$

4 Description of $\mathcal{ST}^x(S_{(g,(P))})$ as $\mathcal{M}^{G(\mathbb{R})_0}(\Gamma_{S_{(g,(P))}})$

4.1 Configurations of triples of points in $\mathbb{P}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$

The goal of this subsection is to prove the following theorem:

$$\mathrm{Conf}_3^{>}(\mathbb{P}^{1|1}) = G(\mathbb{R})_1/\{\pm 1\}$$

Any triple of the form:

$$(\begin{pmatrix} \infty \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix})$$

where $\phi \in G(\mathbb{R})_1$, will be called a *standard triple* of points in $\mathbb{P}^{1|1}$. Let's call ϕ the *odd component* of this standard triple.

The idea is to show that the orbit of any triple $(\begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix})$ of points in $\mathbb{P}^{1|1}$ (such that the orientation of the triple of their bodies $(z_1^\#, z_2^\#, z_3^\#)$ coincides with the standard orientation of $\mathbb{RP}^1$) has only two standard triples in it, and that the odd component of one of this triple is the opposite of the odd component of the other.

Lemma 4.1. *The only matrices in* $\mathrm{SpO}(2|1)(\mathbb{R})$ *sending a standard triple to a standard triple are:*

$$\begin{pmatrix} \pm Id_2 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof. Let:

$$M = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$$

be such a matrix. M stabilises $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} \infty \\ 0 \end{pmatrix}$ thus:

$$\begin{cases} b = 0 \\ \beta = 0 \\ c = 0 \\ \alpha = 0 \end{cases}$$

Hence the equations defining $\mathrm{SpO}(2|1)(\mathbb{R})$ have become:

$$\begin{cases} ad = 1 \\ a\delta = 0 \\ -d\gamma = 0 \\ e^2 + 2\gamma\delta = 1 \end{cases}$$

Thus $\delta = 0$ and $\gamma = 0$. It gives $e^2 = 1$. Because M sends some triple with odd component ϕ_1 to some triple with odd component ϕ_2, this last equation gives $a = d$ and thus $a^2 = 1$. The requirement $\mathrm{Ber}(M) = 1$ is automatically satisfied as soon as $e = 1$, hence the two possible matrices. □

Lemma 4.2. *There exists a matrix M such that:*

$$M \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \infty \\ 0 \end{pmatrix}, \; M \cdot \begin{pmatrix} \infty \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ \psi \end{pmatrix}, \; M \cdot \begin{pmatrix} -1 \\ \phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if and only if $\phi = \pm\psi \in G(\mathbb{R})_1$. Moreover, if the latter condition is satisfied, the are exactly two such matrices, and they are given by:

$$M = \begin{pmatrix} & I_\phi & \\ \phi & 0 & 1 \end{pmatrix}$$

with:

$$I_\phi = \pm \begin{pmatrix} 1 & 1 & \phi \\ -1 & 0 & 0 \end{pmatrix}$$

Proof. The left-most equation gives $d = 0$ and $\beta = 0$, the right-most one gives $b = a - \gamma\phi$ and $\alpha = e\phi$ and the one in the middle $c = -a$. The definition of $\mathrm{SpO}(2|1)(\mathbb{R})$ gives:

$$\begin{cases} (a-\gamma\phi)a = 1 \\ a\delta + a\gamma - e^2\phi = 0 \\ (a-\gamma\phi)\delta = 0 \end{cases}$$

But:

$$\mathrm{Ber}(M) = -\frac{(a+\delta\phi)(a-\gamma\phi)}{e}$$

hence it cannot equals 1 except if $a^\# \neq 0$. Thus the system is equivalent to:

$$\begin{cases} (a-\gamma\phi)a = 1 \\ a\gamma - e^2\phi = 0 \\ \delta = 0 \end{cases}$$

hence $e^2 = 1$, and:

$$\begin{cases} (a-\gamma\phi)a = 1 \\ \gamma = \frac{\phi}{a} \\ \delta = 0 \end{cases}$$

which eventually gives $a^2 = 1$. $\mathrm{Ber}(B) = 1$ as soon as $e = 1$, hence the two matrices. The expression of these matrices implies that $\psi = \pm\phi$. □

Theorem 4.3. *Let:*

$$(\begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix}, \begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix})$$

be a triple of points in $\mathbb{P}^{1|1}$ such that the natural orientation of the triple of their bodies $(z_1^\#, z_2^\#, z_3^\#)$ coincides with the standard orientation on $\mathbb{RP}^1$. Then there exists exactly two matrices $B \in \mathrm{SpO}(2|1)(\mathbb{R})$ such that:

$$B \cdot \begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \infty \\ 0 \end{pmatrix},\ B \cdot \begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -1 \\ \phi \end{pmatrix},\ B \cdot \begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Proof. The first step is the use the constraints written explicitly in the wording of the theorem to get:

$$\begin{cases} \alpha = e\frac{\phi_1-\phi_3}{z_3-z_1} \\ \beta = e\frac{\phi_3 z_1-\phi_1 z_3}{z_3-z_1} \\ b = -az_3 - \gamma\phi_3 \\ c = \frac{az_{23}+\gamma(\phi_2-\phi_3)+\delta(\phi_2-\phi_1)}{z_{12}} \\ d = \frac{az_{23}z_1+\gamma(\phi_2-\phi_3)z_1+\delta(\phi_2 z_1-\phi_1 z_2)}{z_{21}} \end{cases}$$

where we wrote z_{ij} instead of $z_i - z_j$. Then, one uses the structure equation $a\delta - c\gamma - e\alpha = 0$ to obtain:

$$\delta = (\frac{\phi_1\phi_2 + \phi_2\phi_3 + \phi_3\phi_1}{a^2 z_{12} z_{31}} + \frac{z_{23}}{z_{12}})\gamma - \frac{\phi_3 - \phi_1}{a z_{31}}$$

Equation $b\delta - d\gamma - e\beta = 0$ together with the expressions we already have gives:

$$\gamma = \phi_3 \frac{z_{12}}{a z_{23} z_{31}} - \frac{\phi_1\phi_2\phi_3 z_{12}}{a^3 z_{23}^2 z_{31}^2}$$

and that implies:

$$\delta = \frac{\phi_1}{a z_{31}}$$

and:

$$\gamma\delta = \frac{\phi_3\phi_1 z_{12}}{a^2 z_{23} z_{31}^2}$$

The equality $ad - bc - \alpha\beta = 1$ gives the value of a:

$$a = \pm a^{\#}(1 + \frac{1}{2}(\frac{\phi_3\phi_1}{z_{31}} + \frac{\phi_2\phi_3}{z_{23}} + \frac{\phi_1\phi_2}{z_{21}}))$$

with:

$$a^{\#} = \pm\sqrt{\frac{z_{12}}{z_{23}z_{31}}}$$

The last $\mathrm{SpO}(2|1)(\mathbb{R})$ equation $e^2 + 2\gamma\delta = 1$ fixes:

$$e = \pm(1 - \frac{\phi_3\phi_1}{z_{31}})$$

Yet one of the two values of e is not allowed since:

$$\mathrm{Ber}(M) = 1 \Leftrightarrow 1 - e^2\alpha\beta = e (\Rightarrow \left\{ \begin{array}{c} \alpha = e\alpha \\ \beta = e\beta \end{array} \right.) \Rightarrow e = 1 - \alpha\beta = 1 - \frac{\phi_1\phi_3}{z_{31}}$$

The second implication is obtained by multiplying both terms of the preceding equation by the odd numbers α or β. A few more lines of calculus give:

$$(\alpha z_2 + \beta + e\phi_2) z_{31} = \phi_1 z_{23} + \phi_2 z_{31} + \phi_3 z_{12}$$

and:

$$(c z_2 + d + \delta\phi_2) z_{31} = \pm\sqrt{z_{12}z_{23}z_{31}}(1 + \frac{1}{2}(-\frac{\phi_1\phi_2}{z_{12}} - \frac{\phi_2\phi_3}{z_{23}} + \frac{\phi_3\phi_1}{z_{31}}))$$

Hence we have derived the following expression for the odd invariant of a triple of points in $\mathbb{P}^{1|1}$:

$$\boxed{\phi = \pm\frac{\phi_1 z_{23} + \phi_2 z_{31} + \phi_3 z_{12} - \frac{1}{2}\phi_1\phi_2\phi_3}{\sqrt{z_{12}z_{23}z_{31}}}}$$

But:

$$< 1,2 >< 2,3 >< 3,1 >= z_{12}z_{23}z_{31}(1 - \frac{\phi_1\phi_2}{z_{12}} - \frac{\phi_2\phi_3}{z_{23}} - \frac{\phi_3\phi_1}{z_{31}})$$

with:

$$< i,j > \equiv < \left(\begin{array}{c} z_i \\ 1 \\ \phi_i \end{array} \right), \left(\begin{array}{c} z_j \\ 1 \\ \phi_j \end{array} \right) >$$

which gives:

$$(1 + \frac{\phi_1\phi_2}{z_{12}} + \frac{\phi_2\phi_3}{z_{23}} + \frac{\phi_3\phi_1}{z_{31}}) < 1,2 >< 2,3 >< 3,1 >= z_{12}z_{23}z_{31}$$

$$(1+\frac{\phi_1\phi_2}{2z_{12}}+\frac{\phi_2\phi_3}{2z_{23}}+\frac{\phi_3\phi_1}{2z_{31}})\sqrt{<1,2><2,3><3,1>}=\sqrt{z_{12}z_{23}z_{31}}$$

$$\frac{1}{\sqrt{z_{12}z_{23}z_{31}}}=(1-\frac{\phi_1\phi_2}{2z_{12}}-\frac{\phi_2\phi_3}{2z_{23}}-\frac{\phi_3\phi_1}{2z_{31}})\frac{1}{\sqrt{<1,2><2,3><3,1>}}$$

Developing the numerator of ϕ and factorising the terms one gets to:

$$\boxed{\phi=\pm\frac{\phi_1<2,3>+\phi_2<3,1>+\phi_3<1,2>+\phi_1\phi_2\phi_3}{\sqrt{<1,2><2,3><3,1>}}}$$

□

4.2 Properties of the odd invariant of a triple of points in $\mathbb{P}^{1|1}$

Let $(\begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix},\begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix},\begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix})$ be *any* triple in $\mathbb{P}^{1|1}$. The orientation of the "body" triple $(z_1^\#,z_2^\#,z_3^\#)$ can be either positive or negative.

1. If it is positive, we have seen that there exists a matrix $M\in \mathrm{SpO}(2|1)(\mathbb{R})$ sending the initial triple to a standard triple with odd component:

$$\phi=\pm\frac{\phi_1<2,3>+\phi_2<3,1>+\phi_3<1,2>+\phi_1\phi_2\phi_3}{\sqrt{<1,2><2,3><3,1>}}$$

2. If it is negative, then the triple $(z_1^\#,z_3^\#,z_2^\#)$ is positively oriented hence there exists $M\in$ $\mathrm{SpO}(2|1)(\mathbb{R})$ sending the initial triple to $(\begin{pmatrix} \infty \\ 0 \end{pmatrix},\begin{pmatrix} 0 \\ 0 \end{pmatrix},\begin{pmatrix} -1 \\ \phi \end{pmatrix})$ with odd component:

$$\phi=\pm\frac{\phi_1<3,2>+\phi_3<2,1>+\phi_2<1,3>+\phi_1\phi_3\phi_2}{\sqrt{<2,1><3,2><1,3>}}$$

$$\phi=\mp\frac{\phi_1<2,3>+\phi_2<3,1>+\phi_3<1,2>+\phi_1\phi_2\phi_3}{\sqrt{<2,1><3,2><1,3>}}$$

Hence the following definition:

Definition 67. *Let* $(\begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix},\begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix},\begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix})$ *be any triple of points in* $\mathbb{P}^{1|1}$*. The odd invariant* ϕ *of this triple is given by:*

$$\phi=\frac{\phi_1<2,3>+\phi_2<3,1>+\phi_3<1,2>+\phi_1\phi_2\phi_3}{\sqrt{|<1,2><2,3><3,1>|}}\in G(\mathbb{R})_1/\{\pm 1\}$$

Proposition 4.4. *The odd invariant* ϕ *of a triple of points in* $\mathbb{P}^{1|1}$ *is invariant under any permutation of the points.*

The proof of this proposition is trivial, yet it allows us to define the odd-invariant ϕ of any 3-set of points in $\mathbb{P}^{1|1}$ by setting:

$$\phi(\{\begin{pmatrix} z_1 \\ \phi_1 \end{pmatrix},\begin{pmatrix} z_2 \\ \phi_2 \end{pmatrix},\begin{pmatrix} z_3 \\ \phi_3 \end{pmatrix}\})=\frac{\phi_1<2,3>+\phi_2<3,1>+\phi_3<1,2>+\phi_1\phi_2\phi_3}{\sqrt{|<1,2><2,3><3,1>|}}\in G(\mathbb{R})_1/\{\pm 1\}$$

Proposition 4.5. *The numerator of the odd invariant is invariant under the action of* $\mathrm{SpO}(2|1)(\mathbb{R})$.

Proof. We already have an abstract proof of this fact. However, now that we have an explicit expression for the odd invariant it is possible to prove it explicitly. The first step is to use the equations defining $\mathrm{SpO}(2|1)(\mathbb{R})$. Let

$$B = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix} \in \mathrm{SpO}(2|1)(\mathbb{R})$$

and let us assume that a is invertible in $G(\mathbb{R})_0$. First:

$$a\delta - c\gamma - e\alpha = 0$$

implies that:

$$\delta = \frac{c\gamma + e\alpha}{a}$$

Together with $b\delta - d\gamma - e\beta = 0$, one has:

$$\gamma = \alpha b - \beta a$$

thus:

$$\gamma\delta = e\alpha\beta$$

Since $ad - bc = 1 + \alpha\beta$, one has:

$$e\mathrm{Ber}B = (1 + \alpha\beta + \frac{1}{e}(\alpha\beta + e\alpha\beta))$$

and $\mathrm{Ber}B = 1$ which gives:

$$e^2 = \alpha\beta + e(1 + 2\alpha\beta)$$

hence $e^2\alpha = e\alpha$ and $e^2\beta = e\beta$. Now, $e^2 + 2\gamma\delta = e^2 + 2e\alpha\beta = 1$ thus:

$$e = 1 - \alpha\beta$$

giving $e\alpha = \alpha$ and $e\beta = \beta$. Now consider three points in $\mathbb{R}^{2|1}$ denoted by $1 = (x_1, y_1, \phi_1), 2 = (x_2, y_2, \phi_2), 3 = (x_3, y_3, \phi_3)$ and their image by a matrix $B \in \mathrm{SpO}(2|1)(\mathbb{R})$. Their images will be denoted by $1', 2', 3'$. If ϕ_n is the numerator of the odd invariant of the first triple of points, and if ϕ'_n is the numerator of the odd invariant of their image:

$$\phi'_n = \phi'_1 < 2', 3' > +\phi'_2 < 3', 1' > +\phi'_3 < 1', 2' > +\phi'_1\phi'_2\phi'_3$$

$$\phi'_n = \phi'_1 < 2, 3 > +\phi'_2 < 3, 1 > +\phi'_3 < 1, 2 > +\phi'_1\phi'_2\phi'_3$$

$$\phi'_n = (\alpha x_1+\beta y_1+e\phi_1) < 2, 3 > +(\alpha x_2+\beta y_2+e\phi_2) < 3, 1 > +(\alpha x_3+\beta y_3+e\phi_3) < 1, 2 > +(\alpha x_1+\beta y_1+e\phi_1)(\alpha x_2+\beta y_2$$

$$\phi'_n = (e + \alpha\beta)(\phi_1(x_2y_3 - x_3y_2) + \phi_2(x_3y_1 - x_1y_3) + \phi_3(x_1y_2 - x_2y_1)) + (e^3 - 3e)\phi_1\phi_2\phi_3$$

$$\phi'_n = (\phi_1(x_2y_3 - x_3y_2) + \phi_2(x_3y_1 - x_1y_3) + \phi_3(x_1y_2 - x_2y_1)) - 2\phi_1\phi_2\phi_3$$

$$\phi'_n = \phi_n$$

and that concludes the proof. □

The denominator of the odd invariant is obviously invariant under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$, hence we have an explicit proof of the invariance of ϕ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$.

4.3 Refinement of the odd invariant

The odd invariant of a triangle in $\mathbb{S}^{1|1}$ We have seen that the only matrices that sends a standard triple of points in $\mathbb{P}^{1|1}$ to another standard triple are the identity in $\mathrm{SpO}(2|1)(\mathbb{R})$ and:

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SpO}(2|1)(\mathbb{R})$$

Under the action of the latter, the odd component of the point which reduction is -1 becomes its opposite, hence the odd invariant of a positively oriented triple of points in $\mathbb{P}^{1|1}$ is only defined up to its sign. However, we have the following theorem:

Theorem 4.6. *The is a bijection between $G(\mathbb{R})^1$ and the configuration space $\mathrm{Conf}_3^{>}(\mathbb{S}^{1|1})$ which is the configuration space (i.e the space of orbits) of positively oriented triple of distinct points in $\mathbb{S}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$.*

Let's give some definitions first:

Definition 68. *Let $(G(\mathbb{R})_0^{\times})_+$ is the group of invertible elements of $G(\mathbb{R})$ with strictly positive reduction. The space $\mathbb{R}^{2|1}/(G(\mathbb{R})_0^{\times})_+$ is denoted by $\mathbb{S}^{1|1}$. It is a double cover of $\mathbb{RP}^1$ and the orientation of the latter induces a standard orientation on $\mathbb{S}^{1|1}$. Let π be the natural projection $\pi : \mathbb{S}^{1|1} \to \mathbb{P}^{1|1}$. The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{S}^{1|1} & \xrightarrow{\#} & \mathbb{S}^1 \\ \downarrow & & \downarrow \\ \mathbb{P}^{1|1} & \xrightarrow{\#} & \mathbb{RP}^1 \end{array}$$

where the right-most vertical map is the standard covering map.

Definition 69. *A triple of distinct points in $\mathbb{S}^{1|1}$ is said to be positively oriented if the natural orientation of its image by the covering map coincides with the standard orientation of $\mathbb{P}^{1|1}$. A triangle in $\mathbb{P}^{1|1}$ is a positively oriented triple of distinct points in $\mathbb{P}^{1|1}$ up to any circular permutation. The triangle T corresponding to the positively oriented triple of distinct points (A_1, A_2, A_3) will be equivalently denoted by:*

$$T = [A_1, A_2, A_3] = [A_2, A_3, A_1] = [A_3, A_1, A_2]$$

Definition 70. *The action of $\mathrm{SpO}(2|1)(\mathbb{R})$ on $\mathbb{R}^{2|1}$ descends to an action of $\mathrm{SpO}(2|1)(\mathbb{R})$ on $\mathbb{S}^{1|1}$, and $\mathrm{SpO}(2|1)(\mathbb{R})$ acts diagonally on tuples of points in $\mathbb{S}^{1|1}$. Two tuples T_1 and T_2 are said to be equivalent if there is a matrix in $\mathrm{SpO}(2|1)(\mathbb{R})$ sending T_1 to T_2.*

Remark 16. *The function:*

$$\begin{array}{cccc} \sigma : & (\mathbb{R}^{2|1})^2 & \to & \{\pm 1\} \\ & (a,b) & \mapsto & \frac{\det\left(a^{\#}, b^{\#}\right)}{|\det\left(a^{\#}, b^{\#}\right)|} \end{array}$$

is invariant under the action of $((G(\mathbb{R})_0^{\times})_+)^2$ on $(\mathbb{R}^{2|1})^2$ hence is well defined on $(\mathbb{S}^{1|1})^2$.

Lemma 4.7. *Let $T = (A_1, A_2, A_3)$ be a triangle in $\mathbb{S}^{1|1}$. σ defines a map:*

$$\sigma_T : \{(A_1, A_2), (A_2, A_3), (A_3, A_1)\} \to \{\pm 1\}$$

The pre-image of -1 by σ_T contains an even number of elements.

Proof. Let $(\pi(A_1), \pi(A_2), \pi(A_3))$ be the corresponding triple in $\mathbb{P}^{1|1}$ and $\forall i \in [|1,3|]$, let $z_i = \pi(A_i)^{\#}$. Since $(\pi(A_1), \pi(A_2), \pi(A_3))$ is positively oriented, exactly one of the $\{(z_2 - z_1), (z_3 - z_2), (z_1 - z_3)\}$ is negative. Let $T' = [A'_1, A'_2, A'_3]$ be the triangle in $\mathbb{S}^{1|1}$ such that $A'_i = (z_i, 1, \phi_i)$. Then the lemma holds for T'. Eventually note that any lift of $[\pi(A_1), \pi(A_2), \pi(A_3)]$ can be written as $[\pm A'_1, \pm A'_2, \pm A'_3]$, and that multiplication of one of the A'_i by (-1) changes the cardinal of $\sigma_T^{-1}(\{-1\})$ by an even number of elements. That concludes the proof. □

Note that for any $M \in \mathrm{SpO}(2|1)(\mathbb{R})$ and for any $A, B \in \mathbb{S}^{1|1}$, $\sigma(A, B) = \sigma(M \cdot A, M \cdot B)$.

Lemma 4.8. *Every positively oriented triple of points (A_1, A_2, A_3) in $\mathbb{S}^{1|1}$ is equivalent to a unique triple of the form:*

$$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}\right)$$

Such a triple will be called standard.

Proof. According to the last lemma, and without loss of generality one can suppose that $\sigma(A_3, A_1) = 1$. Let $(a_1, a_2, a_3) = (\pi(A_1), \pi(A_2), \pi(A_3)) \in (\mathbb{P}^{1|1})^3$. We know that there exists $M \in \mathrm{SpO}(2|1)(\mathbb{R})$ such that (a_1, a_2, a_3) is sent to a triple of the form:

$$\left(\begin{pmatrix} \infty \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

If:

$$M \cdot A_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

then the assumption on the sign of $\sigma(A_3, A_1)$ implies that:

$$M \cdot A_3 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$$

and $M \cdot A_2$ is completely determined by the fact that:

$$\pi(M \cdot A_2) = \begin{pmatrix} -1 \\ \phi \end{pmatrix}$$

and by the sign of $\sigma(A_1, A_2)$. Otherwise, that is if:

$$M \cdot A_1 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$$

multiplication by the non-trivial matrix in $\mathrm{SpO}(2|1)(\mathbb{R})$ which sends standard triples to standard triples gives the first situation described above. □

Remark 17.
- *If $\sigma_T^{-1}(\{-1\})$ contains only one element, there is a single equivalence to a standard triple.*

- *If $\sigma_T^{-1}(\{-1\})$ contains three elements, one has to choose a vertex of the triangle to be sent to $(1,0,0)$. The matrix:*

$$\begin{pmatrix} -1 & -1 & -\phi \\ 1 & 0 & 0 \\ -\phi & 0 & 1 \end{pmatrix} \in \mathrm{SpO}(2|1)(\mathbb{R})$$

sends the triple:

$$(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix})$$

to:

$$(\begin{pmatrix} 1 \\ -1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix})$$

Hence it allows the following definition of the odd invariant of a triangle in $\mathbb{S}^{1|1}$.

Definition 71. *Any triangle T in $\mathbb{S}^{1|1}$ is equivalent to a triangle of the form:*

$$[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ \mp 1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}]$$

The odd invariant of T is defined to be ϕ in any case. The preceding lemmas and the last remark show that this invariant is well defined, and hence we have proven the theorem of the beginning of this paragraph.

Now this definition can be used to refine the odd invariant of a positively oriented triple of points in $\mathbb{P}^{1|1}$.

Kasteleyn orientations

Definition 72. *Let $(A_1, A_2, A_3) \in (\mathbb{P}^{1|1})^3$ be a positively oriented triple of distinct points, and T be the triangular graph with vertices the A_i. The orientation on $\mathbb{RP}^1$ induces an orientation on T. A Kasteleyn orientation o on T is an orientation on T such that the number of edges on which o coincides with the induced one is even.*

Let $(\tilde{A_1}, \tilde{A_2}, \tilde{A_3})$ be a lift of (A_1, A_2, A_3). Define an orientation on (A_1, A_2, A_3) by orienting the edge $\{A_i, A_j\}$ from A_i to A_j if $\sigma(A_i, A_j) = -1$, and from A_j to A_i otherwise. Then by lemma 4.7 this orientation is Kasteleyn. Note that the lift $(-\tilde{A_1}, -\tilde{A_2}, -\tilde{A_3})$ gives rise to the same Kasteleyn orientation. Conversely, given a Kasteleyn orientation o on the graph underlying the triple (A_1, A_2, A_3), impose $\sigma(A_i, A_j) = -1$ if the edge $\{A_i, A_j\}$ is oriented from A_i to A_j, and $\sigma(A_i, A_j) = 1$ otherwise. Hence o defines two lifts L and L' of (A_1, A_2, A_3) such that $L = -L'$. Thus one has the following proposition:

Proposition 4.9. *There is a one-to-one correspondence between the set of Kasteleyn orientations on $(A_1, A_2, A_3) \in (\mathbb{P}^{1|1})^3$ and $\pm$-classes of lifts of (A_1, A_2, A_3) to $\mathbb{S}^{1|1}$.*

Remark 18. *Let (T, o) be a triangle in $\mathbb{P}^{1|1}$ together with a Kasteleyn orientation o. This Kasteleyn orientation defines two lifts $T_1 = -T_2$ of T in $\mathbb{S}^{1|1}$. These two lifts have opposite odd invariants. Hence the data of a Kasteleyn orientation on T allows us to assign a number $\phi \in G(\mathbb{R})_1$ to any lift of T compatible with the Kasteleyn orientation. Since this invariant is well-defined on orbits of triangles in $\mathbb{S}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$, it is also well-defined on the orbits of the natural action of the same group on the set $\{(T, o)\}$.*

Definition 73. *A switch at a vertex A of a triangle is an automorphism of the set of Kasteleyn orientations on this triangle, which corresponds to reversing the orientation of all edges incident to A. In terms of lifts, it corresponds to a change of leaf of the covering $\mathbb{S}^{1|1} \to \mathbb{P}^{1|1}$ at one vertex.*

Lemma 4.10. *Let (T, o) be a triangle in $\mathbb{P}^{1|1}$ together with a Kasteleyn orientation o. Let $\zeta(T, o)$ be the odd invariant of (T, o). Then for any vertex A of T, let o_A be the Kasteleyn orientation obtained by a switch at A.*

$$\forall A, \zeta(T, o_A) = -\zeta(T, o)$$

Proof. Given the standard triangle in $\mathbb{P}^{1|1}$ of vertices $(\infty, 0)$, $(-1, \phi)$ and $(0, 0)$, it is straightforward to compute the matrix I_ϕ corresponding to the elementary circular permutation of the vertices in the counterclockwise direction: Studying the action of I_ϕ on $\mathbb{S}^{1|1}$, one sees the following fact. If one starts from a triangle in $\mathbb{S}^{1|1}$ in standard form and with odd invariant ϕ and makes a switch in one of the vertices, then:

- Either the new triangle is still in standard form, in which case it is obvious that the new invariant is $-\phi$
- Either it is possible to put the triangle in standard form by acting by $I_{-\phi}$
- Either it is possible to put the triangle in standard form by acting by I_ϕ^{-1}

In all cases one can directly observe the claim. □

4.4 A geometric description of the odd invariant of a triple of points in $\mathbb{P}^{1|1}$

Let $\mathcal{E}$ be the supermanifold consisting of all triples of points $(p_1, p_2, p_3) \in \mathbb{R}^{2|1}$ such that their reductions are non-zero and pairwise non-collinear. Equivalently, $\mathcal{E}$ is the supermanifold consisting of all triples of points $(p_1, p_2, p_3) \in \mathbb{R}^{2|1}$ such that $< i, j >^{\#} \neq 0$ for $i \neq j \in [|1, 3|]$. $\mathcal{E}$ is birationally equivalent to $\mathbb{R}^{6|3}$.

Remark 19. *$\mathcal{E}$ is by definition the domain of two different group actions: the action of* $\mathrm{SpO}(2|1)(\mathbb{R})$ *and the action of $(G(\mathbb{R})_0^\times)_+$. The orbits under the action of $(G(\mathbb{R})_0^\times)_+$ are the points of $\mathbb{S}^{1|1}$.*

Remark 20. *One has a natural map:*

$$\begin{array}{ccc} \mathcal{E} & \to & \mathbb{R}^{2|1} \\ (e_1, e_2, e_3) & \mapsto & e_1 + e_2 + e_3 \end{array}$$

which corresponds to the geometric sum in $\mathbb{R}^{2|1}$.

In the following we will write $e_i = (x_i, y_i, \phi_i)$. We would like to have some geometric interpretation of the bilinear form on $\mathbb{R}^{2|1}$, as the determinant in $\mathbb{R}^2$ is used to construct the kernel of the natural map corresponding to vector sum in the purely even algebraic variety of triple of pairwise non-collinear vectors in $\mathbb{R}^2$. Hence we search three functions $f_1, f_2, f_3 \in \mathcal{O}_\mathcal{E} \simeq \mathcal{C}^\infty(\mathbb{R}^6) \otimes \bigwedge(\phi_1, \phi_2, \phi_3)$ such that:

$$f_1(e_1, e_2, e_3)e_1 + f_2(e_1, e_2, e_3)e_2 + f_3(e_1, e_2, e_3)e_3 = 0 \in \mathbb{R}^{2|1}$$

Remark 21. *First note that if all the f_i are even functions and if for all $i \in [|1, 3|]$, $f_i^{\#} \neq 0$, then:*

$$f_1(e_1, e_2, e_3)e_1 + f_2(e_1, e_2, e_3)e_2 + f_3(e_1, e_2, e_3)e_3 = \begin{pmatrix} * \\ * \\ \sum_{i=1}^{3} f_i^{\#} \phi_i + c\phi_1\phi_2\phi_3 \end{pmatrix}$$

making it impossible to be equal to the zero element of $\mathbb{R}^{2|1}$. However the odd component of the sum is easily sent to zero by multiplication of the result of the sum by its odd component.

Lemma 4.11. *Let $a, b, c \in \mathbb{R}$, and $\phi = a\phi_1 + b\phi_2 + c\phi_3 \in \mathcal{C}^\infty(\mathbb{R}^6) \otimes \bigwedge^1(\phi_1, \phi_2, \phi_3)$. For any $a', b', c' \in \mathbb{R}$:*

$$(a'\phi_1 + b'\phi_2 + c'\phi_3)\phi = 0 \Leftrightarrow (a', b', c') = l \cdot (a, b, c)$$

for some $l \in \mathbb{R}$.

Proof.

$$(a'\phi_1 + b'\phi_2 + c'\phi_3)\phi = (a'b - b'a)\phi_1\phi_2 + (b'c - c'b)\phi_2\phi_3 + (c'a - a'c)\phi_3\phi_1$$

□

Lemma 4.12. *Let $a, b, c \in \mathbb{R}$, and $\phi = a\phi_1 + b\phi_2 + c\phi_3 \in \mathcal{C}^\infty(\mathbb{R}^6) \otimes \bigwedge^1(\phi_1, \phi_2, \phi_3)$. For any $h_{12}, h_{23}, h_{31} \in \mathbb{R}$:*

$$(h_{12}\phi_1\phi_2 + h_{23}\phi_2\phi_3 + h_{31}\phi_3\phi_1)\phi = 0 \Leftrightarrow (h_{12}c + h_{23}a + h_{31}b) = 0$$

Proof.

$$(h_{12}\phi_1\phi_2 + h_{23}\phi_2\phi_3 + h_{31}\phi_3\phi_1)\phi = (h_{12}c + h_{23}a + h_{31}b)\phi_1\phi_2\phi_3$$

□

Remark 22. *One can reformulate the two last lemmas by saying that the right Koszul complex:*

$$0 \to \mathbb{R} \to \bigoplus_{i=1}^{3} \mathbb{R} \cdot \phi_i \to \bigoplus_{i=1}^{3} \mathbb{R} \cdot (\phi_i\phi_{i+1}) \to \mathbb{R} \cdot (\phi_1\phi_2\phi_3) \to 0$$

where $\phi_4 := \phi_1$, and where differential is multiplication by ϕ, is a right resolution of $\mathbb{R}$.

We are interested in functions f_i such that:

$$(\sum_{i=1}^{3} f_i^\# \phi_i)(f_1(e_1, e_2, e_3)e_1 + f_2(e_1, e_2, e_3)e_2 + f_3(e_1, e_2, e_3)e_3) = 0 \in \mathbb{R}^{2|1}$$

which is the same as:

$$(\sum_{i=1}^{3} f_i^\# \phi_i + c\phi_1\phi_2\phi_3)(f_1(e_1, e_2, e_3)e_1 + f_2(e_1, e_2, e_3)e_2 + f_3(e_1, e_2, e_3)e_3) = 0 \in \mathbb{R}^{2|1}$$

for all $c \in \mathbb{R}$. Thus:

$$f_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f_2 \begin{pmatrix} x_2 \\ y_3 \end{pmatrix} + f_3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix}$$

has to be nilpotent. Hence there exists an $\mathbb{R}$-valued section l of $\mathcal{E}$ such that:

$$\begin{cases} f_1^\# = l \cdot \det\left(e_2^\#, e_3^\#\right) \\ f_2^\# = l \cdot \det\left(e_3^\#, e_1^\#\right) \\ f_3^\# = l \cdot \det\left(e_1^\#, e_2^\#\right) \end{cases}$$

and:

$$f_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f_2 \begin{pmatrix} x_2 \\ y_3 \end{pmatrix} + f_3 \begin{pmatrix} x_3 \\ y_3 \end{pmatrix} = \begin{pmatrix} h_{12} \\ g_{12} \end{pmatrix} \phi_1\phi_2 + \begin{pmatrix} h_{23} \\ g_{23} \end{pmatrix} \phi_2\phi_3 + \begin{pmatrix} h_{31} \\ g_{31} \end{pmatrix} \phi_3\phi_1$$

with the h_{ij}, g_{ij} real-valued sections of $\mathcal{O}_\mathcal{E}$. The fact that this sum is annihilated by $(\sum_{i=1}^{3} f_i^\# \phi_i)$ implies:

$$\begin{pmatrix} h_{12} \\ g_{12} \end{pmatrix} \det\left(e_1^\#, e_2^\#\right) + \begin{pmatrix} h_{23} \\ g_{23} \end{pmatrix} \det\left(e_2^\#, e_3^\#\right) + \begin{pmatrix} h_{31} \\ g_{31} \end{pmatrix} \det\left(e_3^\#, e_1^\#\right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

A priori there are lots of f_i satisfying this constraint, but it is possible to choose the f_i such that they are constant under the actions of $\mathrm{SpO}(2|1)(\mathbb{R})$ and $((G(\mathbb{R})_0^\times)_+)^3$.

- The natural choice of $f_1 = l\cdot <2,3>$, $f_2 = l\cdot <3,1>$ and $f_3 = l\cdot <1,2>$, for any $l \in (\mathcal{O}_{\mathcal{E}})^\times$ which is constant on each $\mathrm{SpO}(2|1)(\mathbb{R})$-orbit gives rise to f_i which are also constant on each $\mathrm{SpO}(2|1)(\mathbb{R})$-orbit.
- Then we choose l is order to have *local* f_i, in the following sense. We want each f_i to be invariant under the action of $(G(\mathbb{R})_0^\times)_+$ on each of the e_j such that $j \neq i$. This requirement fixes the value of $l = \sqrt{<1,2><2,3><3,1>}$ and hence of the f_i. For example:

$$f_1 = \sqrt{\frac{<2,3>}{<1,2><3,1>}}$$

Now, let Φ be the map:

$$\begin{array}{ccc} \mathcal{E} & \to & \mathbb{R}^{2|1} \\ (e_1, e_2, e_3) & \mapsto & f_1 e_1 + f_2 e_2 + f_3 e_3 \end{array}$$

Remark 23. *Let $\phi \in G(\mathbb{R})_1$. Let m_ϕ be the map:*

$$\begin{array}{ccc} \mathbb{R}^{2|1} & \to & \mathbb{R}^{1|2} \\ e & \mapsto & \phi \cdot e \end{array}$$

If two sections $\phi_1 \in \mathcal{O}_{\mathcal{E},1}$ and $\phi_2 \in \mathcal{O}_{\mathcal{E},1}$ differ by a term of order 3 is the odd variables, then the kernels of $m_{\phi_1} \circ \Phi$ and $m_{\phi_2} \circ \Phi$ are isomorphic. It defines an equivalence relation on $\mathcal{O}_{\mathcal{E},1}$.

Lemma 4.13. *The odd component of $f_1 e_1 + f_2 e_2 + f_3 e_3$ is the function:*

$$\phi_1 f_1 + \phi_2 f_2 + \phi_3 f_3 \in \mathcal{O}_{\mathcal{E},1}$$

There exists an equivalent section of $\mathcal{O}_{\mathcal{E},1}$ which is constant on the orbits of the action of $\mathrm{SpO}(2|1)(\mathbb{R})$.

Proof. Take the section:

$$\phi_1 f_1 + \phi_2 f_2 + \phi_3 f_3 + \phi_1\phi_2\phi_3 f_1 f_2 f_3 \in \mathcal{O}_{\mathcal{E},1}$$

The results of the last section concludes the proof of the lemma. □

Remark 24. *Interestingly, the requirement of being constant under the action of* $\mathrm{SpO}(2|1)(\mathbb{R})$ *implies that this section is also constant on the orbits of $((G(\mathbb{R})_0^\times)_+)^3$.*

As in the classical case, it is possible to construct some connection on the graph such that equivalence under gauge transformation construct well-defined functions on orbits of the action of $(G(\mathbb{R})_0^\times)_+$.

Graph structure Start with a bipartite graph with one black vertex linked to three white vertices. This graph is denoted by $\Gamma_{(0,(3))}$. To each white vertex assign the $(G(\mathbb{R})_0)_+$-module generated by an element of $\mathbb{R}^{2|1}$ which reduction is non zero, and such that for each pair of white vertices the reductions of their generators are non collinear vectors in $\mathbb{R}^2$. Let e_1, e_2, e_3 be the generators of the three white vertices, and let D_1, D_2, D_3 be the corresponding $(G(\mathbb{R})_0)_+$-modules. In the last subsection we constructed the maps:

$$\bigoplus_{i=1}^3 D_i \xrightarrow{\Phi} \mathbb{R}^{2|1} \xrightarrow{m_\phi} \mathbb{R}^{1|2}$$

where the first one is the geometric sum of elements in $\mathbb{R}^{2|1}$ and where ϕ is the odd invariant of the D_i. The kernel of $m_\phi \circ \Phi$ is a free $(G(\mathbb{R})_0)_+$-module of purely even rank $1|0$. Assign to the black vertex of the graph this free $(G(\mathbb{R})_0)_+$-module as well as the odd invariant ϕ. As in the classical case, one has natural isomorphisms of modules from the module assigned to the black vertex to the modules assigned to the white vertices. As soon as one has chosen a basis for each of the modules, the isomorphisms can be represented in matrix form as a number in $(G(\mathbb{R})_0^\times)_+$. Hence instead of a module assign to each vertex of the graph a generator of the corresponding module. This procedure defines an invertible even number on each edge of the graph, and the data of these numbers form a $(G(\mathbb{R})_0^\times)_+$-connection on the graph.

Remark 25. *Since the connection is of gauge group $(G(\mathbb{R})_0^\times)_+$, the data of the e_i defines a Kasteleyn orientation on the triangle with vertices the e_i. A Kasteleyn orientation on this triangle can be equivalently encoded on the bipartite graph by marking an even number of edges incident to the black vertex. These edges will be dashed on our figures.*

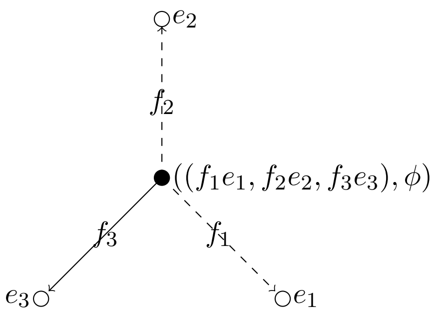


Figure 17: Assignment of numbers on the edges and the black vertex on a graph corresponding to a triple of points in $\mathbb{R}^{2|1}$. The choice of dashed edges is equivalent to saying that $\sigma(e_3^\#, e_1^\#) = \sigma(e_2^\#, e_3^\#) = -1$, and $\sigma(e_1^\#, e_2^\#) = 1$.

Conversely, start with a bipartite graph with one black vertex linked to three white vertices denoted $i = 1..3$, with two dashed edges, and with the following data:

- An odd number ϕ corresponding to the black vertex
- A number in $(G(\mathbb{R})_0^\times)_+$ for each edge, denoted a_i for the edge linking the black vertex to the white vertex i.

Our goal is to find an orbit of triple of points (e_1, e_2, e_3) in $\mathbb{S}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$ corresponding to this structure on the bipartite graph.

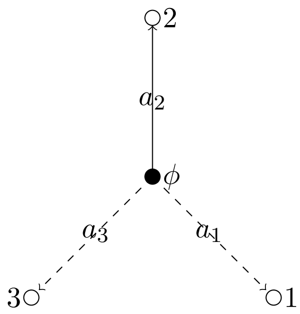


Figure 18: Bipartite graph with a $(G(\mathbb{R})_0^\times)_+$-connection and an odd number associated to its black vertex

The configuration of dashed edges implies that one have to search a representative triple of the $\mathrm{SpO}(2|1)(\mathbb{R})$-orbit of the form:

$$\begin{cases} e_1 = l_1(1,0,0) \\ e_2 = l_2(1,-1,\phi) \\ e_3 = l_3(0,-1,0) \end{cases}$$

where the $l_i \in (G(\mathbb{R})_0^\times)_+$. Hence one is forced to choose $l_i = a_i^{-1}$ for all $i \in [1,3]$, and the data corresponds to a single orbit of triples of points in $\mathbb{S}^{1|1}$ under the action of $\mathrm{SpO}(2|1)(\mathbb{R})$.

Hence we have:

Theorem 4.14.

$$\mathrm{Conf}_3^{>}(\mathbb{S}^{1|1}) \simeq \{o\} \amalg \mathcal{M}^{(G(\mathbb{R})_0^\times)_+}(\Gamma_{(0,(3))}, o) \amalg G(\mathbb{R})_1 \simeq \{o\} \amalg G(\mathbb{R})_1$$

where o is a choice of two edges of the three of $\Gamma_{(0,(3))}$.

4.5 Quadruples of points and flips

Let z_i, $i = 1..4$ be four points in $\mathbb{P}^{1|1}$ such that $(z_1^\#, z_2^\#, z_3^\#)$ and $(z_3^\#, z_4^\#, z_1^\#)$ are positively oriented. Let Γ be the triangulated quadrilateral with edges (e_i, e_{i+1}) and (e_1, e_3). Choose a Kasteleyn orientation on Γ, that is an orientation of the edges of Γ such that for each triangle there are an even number of edges on which this orientation coincides with the one induced by the standard orientation on $\mathbb{P}^{1|1}$, and such that the orientation it induces on the external quadrilateral is again Kasteleyn. As before:

Lemma 4.15. *There is a one-to-one correspondence between the set of Kasteleyn orientations on the quadrilateral of vertices the z_i and the $\pm$-equivalence classes of lifts of the e_i to $\mathbb{S}^{1|1}$. Moreover, given a Kasteleyn orientation on the quadrilateral, there exists a unique Kasteleyn orientation on Γ which coincides with the latter on the external edges.*

Construct a bipartite graph $\Gamma_{(0,(4))}$ from Γ by taking a white vertex for each z_i, a black vertex for each triangle of Γ and edges between a black vertex v_B and a white vertex v_W if v_W is a vertex of the triangle in Γ corresponding to v_B. Now the moduli space of $(G(\mathbb{R})_0^\times)_+$-connections on $\Gamma_{(0,(4))}$ has an invariant given by the monodromy around the single loop. This monodromy is by construction invariant under the actions of $\mathrm{SpO}(2|1)(\mathbb{R})$ and $((G(\mathbb{R})_0^\times)_+)^4$. It is called the super-cross-ratio of the quadruple (z_1, z_2, z_3, z_4), and it is given by:

$$\chi(z_1, z_2, z_3, z_4) = \frac{<1,2><3,4>}{<2,3><4,1>}$$

Hence:

Theorem 4.16.

$$\mathrm{Conf}_4^{>}(\mathbb{S}^{1|1}) \simeq \{o\} \amalg \mathcal{M}^{(G(\mathbb{R})_0^\times)_+}(\Gamma_{(0,(4))}, o) \amalg (G(\mathbb{R})_1)^2$$

Matrix model

- As in the classical case, one can encode the data of the connection in a matrix. Whereas the reduction of each coefficient of the connection is strictly positive, one can take into account the Kasteleyn orientation by adding a minus sign in the matrix each time the corresponding edge is dashed. Each white vertex corresponds to a column of the matrix whereas lines correspond to black vertices.
- One adds a column for the odd invariants

Hence a matrix following those rules corresponds to the data of a Kasteleyn orientation on $\Gamma_{0,(4)}$, a $(G(\mathbb{R})_0^\times)_+$-connection on $\Gamma_{0,(4)}$ as well as two numbers in $G(\mathbb{R})_1$ which are the odd invariants assigned to the two black vertices of $\Gamma_{0,(4)}$. However, in what follows, we will somehow abusively talk about *the connection defined by such a matrix.*

For example, the matrix corresponding to the graph on the left of figure 19 is given by:

$$\begin{pmatrix} -a & b & -c & 0 & \phi \\ f & 0 & -d & -e & \psi \end{pmatrix}$$

As before, left-multiplication by a diagonal matrix corresponds to a change of basis in the modules corresponding to the black vertices except that the last column of the matrix is in the trivial representation, since it already contains gauge invariants. If some connection on $\Gamma_{(0,(4))}$ is represented by the matrix M, and for $\lambda_1, \lambda_2 \in (G(\mathbb{R})_0^\times)_+$, $\mathrm{diag}(\lambda_A, \lambda_B) \cdot M$, which is understood as the result of the multiplication of all columns of the matrix but the last by $\mathrm{diag}(\lambda_A, \lambda_B)$, is the equivalent connection obtained through the gauge transformation with support in the set $\{A, B\}$ and given by $g(A) = \lambda_A$ and $g(B) = \lambda_B$.

Right-multiplication by a matrix of the form $\mathrm{diag}(\lambda_1, ..., \lambda_4, 1)$ corresponds to the gauge transformation with support in the set of the white vertices and defined by $g(v_i) = \lambda_i^{-1}$ for all $i \in [|1, 4|]$.

Changes of coordinates under a flip The last lemma emphasise the isomorphism between Kasteleyn orientation on Γ and Kasteleyn orientation on the quadrilateral with vertices the z_i. We are interested in the change a coordinates in our parametrisation of $\mathrm{Conf}_4^>(\mathbb{S}^{1|1})$ due to a flip. The Kasteleyn orientation on the quadrilateral remains the same under the flip hence if α is the internal edge of Γ, there is a unique Kasteleyn orientation on Γ_α which coincides with the Kasteleyn orientation on Γ on the external quadrilateral.

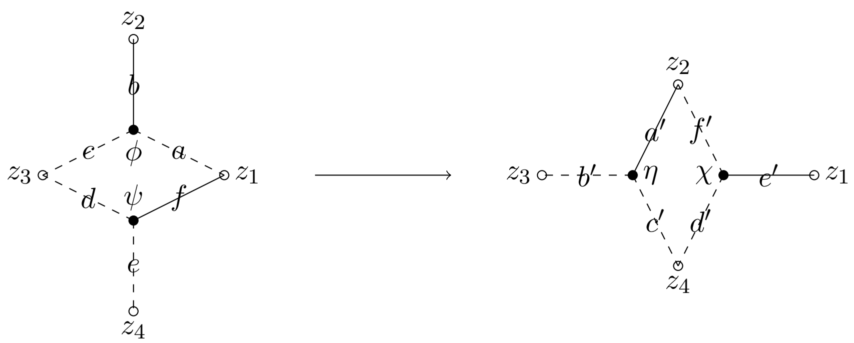


Figure 19: Transformation of a Kasteleyn orientation under a flip.

Proposition 4.17. *If $T = [ABC]$ is a triangle in $\mathbb{S}^{1|1}$, let's denote by $\zeta([ABC])$ its odd invariant. Let $z \in G(\mathbb{R})_0^\times$ such that $z^\# > 0$. Then:*

$$\zeta(\begin{bmatrix} 1 & 0 & z \\ 0 & -1 & 1 \\ 0 & 0 & \sqrt{z}\phi \end{bmatrix}) = -\phi$$

$$\zeta(\begin{bmatrix} 1 & 0 & z \\ -1 & -1 & 1 \\ \phi & 0 & -\sqrt{z}\psi \end{bmatrix}) = \frac{\phi\sqrt{z} + \psi}{\sqrt{1+z}}$$

$$\zeta(\begin{bmatrix} 1 & 0 & z \\ \phi & 0 & -\sqrt{z}\psi \\ 1 & 0 & 0 \end{bmatrix}) = \frac{-\phi + \psi\sqrt{z}}{\sqrt{1+z}}$$

Proof. These formula can be easily obtained by a straightforward calculation. They can also be found in [Bou13]. □

Note that for $z \in G(\mathbb{R})_0^\times$ such that $z^\# > 0$ and for the special connection on $\Gamma_{(0,(4))}$ defined by:

$$\begin{pmatrix} -1 & 1 & -1 & 0 & \phi \\ \sqrt{z} & 0 & -\frac{1}{z} & -\frac{1}{z} & \psi \end{pmatrix}$$

which is the connection defined by the quadruple:

$$(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ \phi \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} z \\ 1 \\ -\sqrt{z}\psi \end{pmatrix})$$

a flip in the internal edge corresponds to left-multiplication by the matrix:

$$\begin{pmatrix} \sqrt{\frac{z}{z+1}} & \frac{1}{\sqrt{z+1}} \\ -\frac{1}{\sqrt{z+1}} & \sqrt{\frac{z}{z+1}} \end{pmatrix}$$

as depicted in the figure 20, except that this time the odd column of the connection matrix is also multiplied by the latter:

$$\begin{cases} \eta = \frac{\phi\sqrt{z}+\psi}{\sqrt{1+x}} \\ \chi = \frac{-\phi+\sqrt{z}\psi}{\sqrt{1+x}} \end{cases}$$

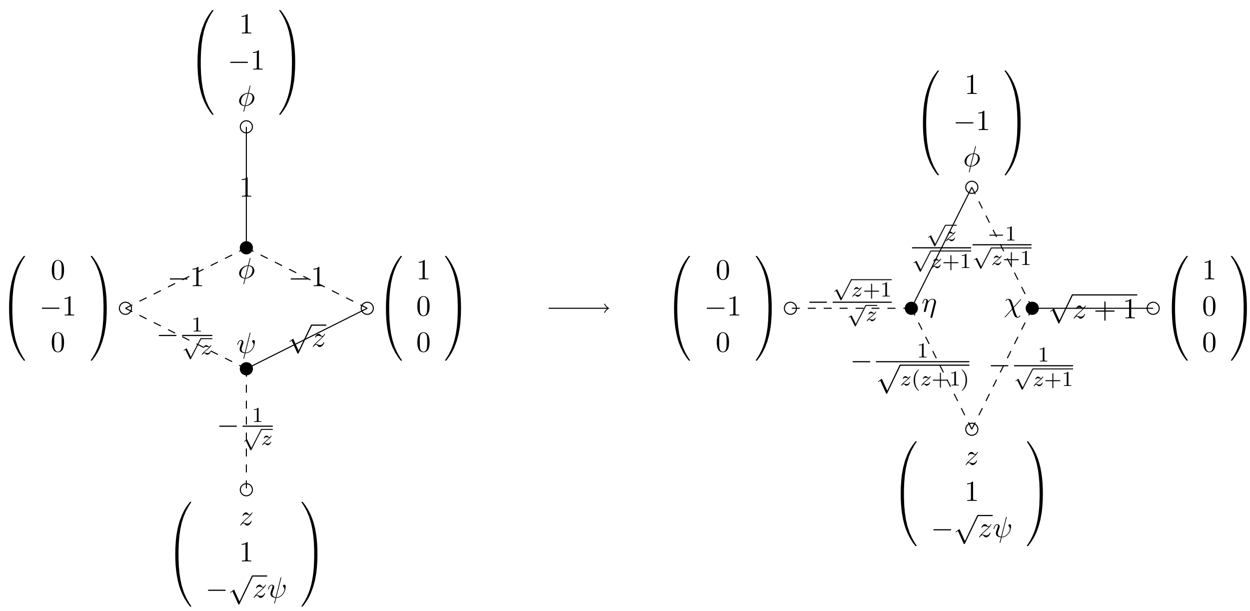


Figure 20: Transformation of the connection induced by a standard quadruple in $\mathbb{S}^{1|1}$ under a flip.

Let:

$$M = \begin{pmatrix} -a & b & -c & 0 & \phi \\ f & 0 & -d & -e & \psi \end{pmatrix}$$

be a matrix representing any connection, and let A be the black vertex corresponding to the first line, and B the one corresponding to the second line. To compute how the coefficient of the connection vary under a flip, consider the equivalent connection given by:

$$\begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \sqrt{\frac{c}{adf}} \end{pmatrix} \cdot M$$

It is in standard form up to the right action by the matrix:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{a}{b} & 0 & 0 & 0 \\ 0 & 0 & \frac{a}{c} & 0 & 0 \\ 0 & 0 & 0 & \frac{da}{ec} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Since it acts on the right, it commutes with the flip hence the flip on this new matrix is also given by:

$$\begin{pmatrix} \sqrt{\frac{z}{z+1}} & \frac{1}{\sqrt{z+1}} \\ -\frac{1}{\sqrt{z+1}} & \sqrt{\frac{z}{z+1}} \end{pmatrix}$$

where $z = \frac{fc}{ad}$. Hence for a general connection given by the matrix:

$$M = \begin{pmatrix} -a & b & -c & 0 & \phi \\ f & 0 & -d & -e & \psi \end{pmatrix}$$

the flip is given by the left multiplication by:

$$F = \begin{pmatrix} \frac{1}{a}\sqrt{\frac{z}{z+1}} & \frac{1}{f}\sqrt{\frac{z}{z+1}} \\ \frac{-1}{a\sqrt{z+1}} & \frac{z}{f\sqrt{z+1}} \end{pmatrix}$$

The last thing one needs to understand the change of coordinates under a transition function from one chart to another is the change a coordinates under a flip.

Changes of coordinates under a switch of the Kasteleyn orientation

Remark 26. *Consider the bipartite graph $\Gamma_{(0,(4))}$ corresponding to Γ, and let v be one of its white vertices. A switch at v corresponds to the following transformation: for each black vertex linked to v, reverse the nature of all edges incident to it except the one linking it to v.*

Let:

$$M = \begin{pmatrix} -a & b & -c & 0 & \phi \\ f & 0 & -d & -e & \psi \end{pmatrix}$$

be a matrix representing any connection on $\Gamma_{(0,(4))}$. A switch in a vertex corresponding to some column C imposes the following transformation. For any line L of M:

- If $L \cap C = 0$, then all coefficients of L remains the same.
- If $L \cap C \neq 0$, then all coefficients of $L \backslash (L \cap C)$ are multiplied by -1, including the odd one.

Example 6. *To apply these methods to a concrete case, take the connection corresponding to the right diagram of the figure 20. After a relabelling of the edges with respect to the one we are used to, one can represent it by the matrix:*

$$M_1 = \begin{pmatrix} \frac{\sqrt{z}}{\sqrt{z+1}} & -\frac{\sqrt{z+1}}{\sqrt{z}} & -\frac{1}{\sqrt{z(z+1)}} & 0 & \eta \\ -\frac{1}{\sqrt{z+1}} & 0 & -\frac{1}{\sqrt{z+1}} & \sqrt{z+1} & \chi \end{pmatrix}$$

and switch the third column to obtain:

$$M_2 = \begin{pmatrix} -\frac{\sqrt{z}}{\sqrt{z+1}} & \frac{\sqrt{z+1}}{\sqrt{z}} & -\frac{1}{\sqrt{z(z+1)}} & 0 & -\eta \\ \frac{1}{\sqrt{z+1}} & 0 & -\frac{1}{\sqrt{z+1}} & -\sqrt{z+1} & -\chi \end{pmatrix}$$

then by an appropriate gauge transformation with support in the set of black vertices we obtain the following matrix:

$$M_3 = \begin{pmatrix} -1 & \frac{z+1}{z} & -\frac{1}{z} & 0 & -\eta \\ \frac{1}{\sqrt{z}} & 0 & -\frac{1}{\sqrt{z}} & -\frac{(z+1)}{\sqrt{z}} & -\chi \end{pmatrix}$$

which is in standard form up to right-multiplication by some diagonal matrix. Here since the monodromy is $\frac{1}{z}$, the flip is given by the left multiplication by:

$$\begin{pmatrix} \sqrt{\frac{1}{z+1}} & \sqrt{\frac{z}{z+1}} \\ -\sqrt{\frac{z}{z+1}} & \sqrt{\frac{1}{z+1}} \end{pmatrix}$$

and one obtains:

$$M_4 = \begin{pmatrix} 0 & \frac{\sqrt{z+1}}{z} & -\frac{\sqrt{z+1}}{z} & -\sqrt{z+1} & -\sqrt{\frac{1}{z+1}}\eta - \sqrt{\frac{z}{z+1}}\chi \\ \sqrt{\frac{z+1}{z}} & -\sqrt{\frac{z+1}{z}} & 0 & -\sqrt{\frac{z+1}{z}} & \sqrt{\frac{z}{z+1}}\eta - \sqrt{\frac{1}{z+1}}\chi \end{pmatrix}$$

One has to switch the third column again, giving:

$$M_5 = \begin{pmatrix} 0 & -\frac{\sqrt{z+1}}{z} & -\frac{\sqrt{z+1}}{z} & \sqrt{z+1} & \sqrt{\frac{1}{z+1}}\eta + \sqrt{\frac{z}{z+1}}\chi \\ \sqrt{\frac{z+1}{z}} & -\sqrt{\frac{z+1}{z}} & 0 & -\sqrt{\frac{z+1}{z}} & \sqrt{\frac{z}{z+1}}\eta - \sqrt{\frac{1}{z+1}}\chi \end{pmatrix}$$

which up to right-multiplication by a diagonal matrix is equal to:

$$M_6 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{z}} & -\frac{1}{\sqrt{z}} & \sqrt{z} & \sqrt{\frac{1}{z+1}}\eta + \sqrt{\frac{z}{z+1}}\chi \\ 1 & -1 & 0 & -1 & \sqrt{\frac{z}{z+1}}\eta - \sqrt{\frac{1}{z+1}}\chi \end{pmatrix}$$

Eventually, computing the odd invariants one finds:

$$M_6 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{z}} & -\frac{1}{\sqrt{z}} & \sqrt{z} & \psi \\ 1 & -1 & 0 & -1 & \phi \end{pmatrix}$$

Action of $\mathfrak{S}_4$ on the super cross-ratio As in the classical case, under permutation of the arguments the super cross-ratio does not take 24 values but merely 6. Let's draw the tetrahedra carrying the values of the super cross-ratio under the permutation of its arguments:

4.6 The super Teichmüller space $\mathcal{ST}^x(S_{(g,(P))})$ of a ciliated surface

Following [Bou13], we now define the super Teichmüller space of a ciliated surface S without cilia.

Definition 74. *Let $S_{g,(P)}$ be a ciliated surface which boundary has s connected components, and without cilia. Let* $\mathrm{ST}(S_{g,(P)})$ *be the set of monomorphisms:*

$$\psi : \pi_1(S_{g,(P)}) \to \mathrm{SpO}(2|1)(\mathbb{R})$$

such that the reduction of $\psi(\pi_1(S_{g,(P)}))$ is a Fuchsian group. $\mathrm{SpO}(2|1)(\mathbb{R})$ *acts on this set by overall conjugation. The super Teichmüller space $\mathcal{ST}(S_{g,(P)})$ of $S_{g,(P)}$ is the quotient:*

$$\mathcal{ST}(S_{g,(P)}) = \mathrm{ST}(S_{g,(P)})/\mathrm{SpO}(2|1)(\mathbb{R})$$

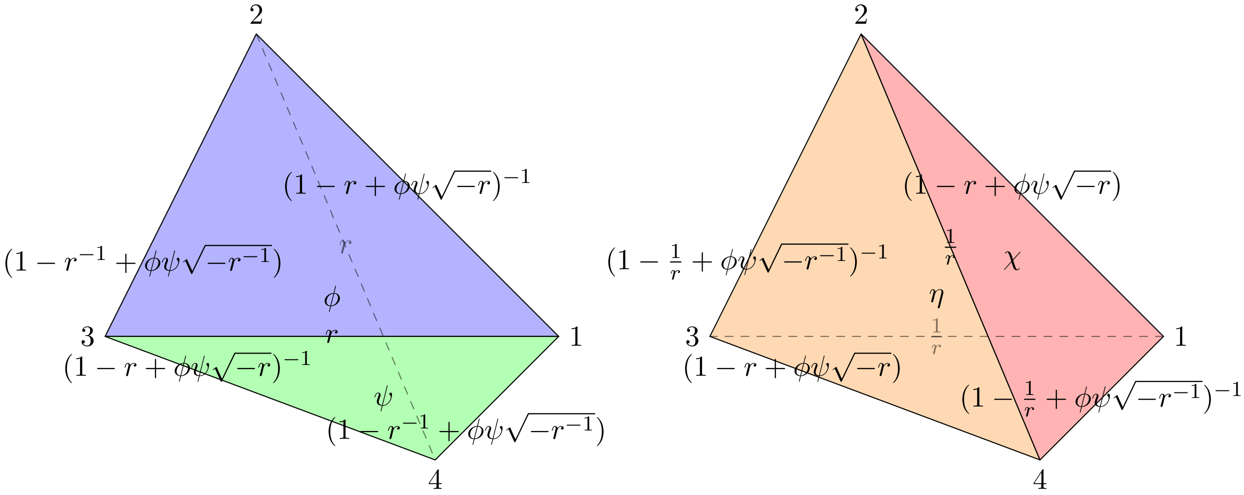


Figure 21: The two tetrahedra, with super cross-ratios on the edges. The one on the left is the tetrahedron with positive orientation, the one on the left is the one with the orientation reversed. Here we have $\eta = \frac{\sqrt{z}\phi+\psi}{\sqrt{1+z}}$ and $\chi = \frac{-\phi+\sqrt{z}\psi}{\sqrt{1+z}}$ which implies $\eta\chi = \phi\psi$.

Remark 27. *The de Witt topology on* $\mathrm{SpO}(2|1)(\mathbb{R})$ *induces a topology on* $\mathcal{ST}(S_{g,(P)})$ *which is also de Witt.*

Moreover we have the following theorem, proven by Natanzon and Hodgkin (see the end of 2. in [Bou13] for references). Fix the nature of the connected components of the boundary of $S_{g,(P)}$: either hole or puncture. Note that each puncture gives rise to an additional constraint coming from the fact that the length of the geodesic corresponding to the reduction of a loop around a puncture is zero. Let h be the number of holes and let p be the number of punctures. For now, we will refer to $S_{g,(P)}$ as $S_{g,k,m}$.

Theorem 4.18. *1.* $\mathcal{ST}(S_{g,h,p})$ *can be written as the quotient of a supermanifold* $\hat{\mathcal{ST}}(S_{g,h,p})$ *by the standard action of* $\mathbb{Z}_2$ *on its odd part.*

2. The space $\hat{\mathcal{ST}}(S_{g,h,p})$ *has several connected components* $\hat{\mathcal{ST}}(S_{g,h,p})^\sigma$*, indexed by the spin structures* σ *on* S*.*

3. For each spin structure σ*,* $\hat{\mathcal{ST}}(S_{g,h,p})^\sigma$ *satisfies:*

$$\hat{\mathcal{ST}}(S_{g,h,p})^\sigma \simeq \mathbb{R}^{6g-6+3h+2p|4g-4+2h+2p-p_R^\sigma}$$

where p_R^σ *is the number of Raymond punctures of the spin structure* σ*.*

Remark 28. *We are not going to discuss properly about spin structures on Riemann surfaces, but we will only give a few definitions to understand the link between Kasteleyn orientations and spin structures. According to [Wit12], Raymond-type punctures corresponds to singularity on the super structure of the surface, hence as in the classical it give rise to an additional constraint, but this time on the odd parameters of this super-structure, explaining why each Raymond-type puncture removes one odd parameter in the point 3. of the last theorem.*

Kasteleyn orientation on surface graph and spin structures Here we reproduce some results of [CR07].

Definition 75. *Let Σ be a connected oriented closed surface. A surface graph in Σ is a graph embedded in Σ as the 1-skeleton of a cellular decomposition X of Σ. Σ induces a fat structure on any surface graph in Σ. Any finite connected graph can be realised as a surface graph, because it is always possible to embed it in some Σ_g for sufficiently large genus g. If g is minimal, the embedding gives a cellular decomposition of Σ_g.*

Definition 76. *The orientation on Σ induces an orientation of any face of a surface graph Γ in Σ. A Kasteleyn orientation K on Γ is an orientation such that for each face f of Γ:*

$$\prod_{e \in \partial f} \epsilon_f^K(e) = -1$$

where $\epsilon_f^K(e) = 1$ if the orientation of e induces by the orientation of f coincides with the orientation $K(e)$, and -1 otherwise.

Definition 77. *A switch at some vertex v of Γ is the transformation of Kasteleyn orientations on Γ obtained by reversing the orientation of all edges of Γ incident to v. Two Kasteleyn orientations which can be related by a sequence of switches are said to be equivalent.*

Now we give three important results from [CR07]. Their proofs can be found in the same paper.

Proposition 4.19. *Let $\Gamma \subset \Sigma$ be a surface graph. Then there exists a Kasteleyn orientation on Γ is and only if Γ has en even number of vertices.*

Theorem 4.20. *Let $\Gamma \subset \Sigma$ be a surface graph with an even number of vertices. Then the set of equivalence classes of Kasteleyn orientations on Γ is an affine $H^1(\sigma, \mathbb{Z}_2)$-space.*

Corollary 4.21. *Let $\Gamma \subset \Sigma$ be a surface graph with an even number of vertices, and such that Σ is of genus g. Then there are exactly 2^{2g} equivalence classes of Kasteleyn orientations on Γ.*

The main fact for our study is the following theorem. It can be found in [CR08].

Theorem 4.22. *The choice of a dimer configuration on a surface graph $\Gamma \subset \Sigma$ induces an isomorphism of affine $H^1(\sigma, \mathbb{Z}_2)$-spaces from the set of equivalence classes of Kasteleyn orientation to the set of spin structures on Σ.*

The application of these theorems to the study of super-Teichmüller spaces is also explicitly done in [Bou13].

Following [Bou13], we define:

Definition 78. *Let $S_{g,(P)}$ be a ciliated surface with a complex structure. Recall that any boundary component of $S_{g,(P)}$ can be of three different types:*

- *Either it carries a non-zero number of cilia*
- *Either it carries no cilia an its boundary is isomorphic to an annulus.In this case it is called a hole.*
- *Either it carries no cilia an its boundary is isomorphic to a punctured disk .In this case it is called a puncture.*

Since any super-Riemann structure gives by reduction a regular complex structure on $S_{g,(P)}$, let's distinguish a priori in the topological $S_{g,(P)}$ the cilia $c_1, ..., c_l$, the holes $h_1, ..., h_k$ and the punctures $p_1, ..., p_m$. Consider the set T of triples of the form:

$$(\rho, o_{i i=1,..,k}, F_{j j=1,..,l,l+1,l+k})$$

where ρ is a morphism $\rho \to \mathrm{SpO}(2|1)(\mathbb{R})$ such that the reduction of its image is a Fuchsian group, $o_{i i=1,..,k}$ is an orientation of each hole, and $F_{j j=1,..,l,l+1,l+k}$ is the distinguished $\pi_1(S_{g,(P)})$-equivariant set consisting of the preimages of cilia and punctures.

Definition 79. *Now define a equivalence relation on T by saying that two triples (ρ, o, F) and (ρ', o', F') are equivalent if there exits $B \in \mathrm{SpO}(2|1)(\mathbb{R})$ such that:*

$$\begin{cases} \rho' = B\rho B^{-1} \\ \forall i \ o'_i = o_i \\ \forall j \ F'_j = B \cdot F_j \end{cases}$$

Definition 80. *The super Teichmüller $\mathcal{X}$-space of $S_{g,(P)}$ is denoted by $\mathcal{ST}^x(S_{(g,(P))})$ and is defined to be the set of equivalence classes in T. It is an H^∞ de Witt supermanifold ([Bou13]).*

A main result of [Bou13] is the following, which is directly extended to ciliated surfaces.

Theorem 4.23. *Let $S_(g,(P))$ be a ciliated surface with triangulation Γ and let K be a Kasteleyn orientation on Γ. The choice of an even number with strictly positive reduction for each internal edge and of an odd number for each face of the triangulation provides a global parametrisation of the connected component of the super Teichmüller $\mathcal{X}$-space indexed by K up to the diagonal action of $\mathbb{Z}_2$ on its odd part.*

Remark 29. *In some cases one has to consider an hexagonalisation associated with the triangulation in order to be able to define Kasteleyn orientations on the surface graph (with boundary). We refer to [Bou13] for those considerations as well as the proof of the last theorem.*

4.7 The super Poisson structure on $\mathcal{ST}^x(S_{(g,(P))})$

Definition 81. *A Poisson super-algebra is a super-algebra A over $\mathbb{R}$ with a Poisson super-bracket:*

$$\{.,.\} : A \times A \to A$$

which satisfies, for all $a, b, c \in A$:

1. $\{a, b\} = (-1)^{\tilde{a}\tilde{b}}\{b, a\}$
2. $\{a, \{b, c\}\} + (-1)^{\tilde{a}(\tilde{b}+\tilde{c})}\{b, \{c, a\}\} + (-1)^{\tilde{z}(\tilde{x}+\tilde{y})}\{c, \{a, b\}\} = 0$
3. $\{x, yz\} = \{x, y\}z + (-1)^{\tilde{x}\tilde{y}} y\{x, z\}$

Definition 82. *A Poisson supermanifold is an supermanifold (algebraic, G^∞ or H^∞) such that the structure sheaf (of algebraic functions, G^∞-supersmooth or H^∞) is equipped with a Poisson super-bracket.*

Definition 83. *$\mathcal{ST}^x(S_{(g,(P))})$ has a global parametrisation by even variables denoted by x_i and odd variables denoted by ζ_i. One defines two odd derivations $\overrightarrow{\partial_i}$ and $\overleftarrow{\partial_i}$ on the ring of regular functions of $\mathcal{ST}^x(S_{(g,(P))})$ which are given by:*

- $\overrightarrow{\partial_i}\zeta_j = \overleftarrow{\partial_i}\zeta_j = \delta_{ij}$
- $\overrightarrow{\partial_i}x_j = \overleftarrow{\partial_i}x_j = 0$
- $\overrightarrow{\partial_i}(\zeta_1 ... \zeta_i ... \zeta_n) = (-1)^{i-1}\zeta_1 ... \hat{\zeta_i} ... \zeta_n$
- $\overleftarrow{\partial_i}(\zeta_1 ... \zeta_i ... \zeta_n) = (-1)^{n-i}\zeta_1 ... \hat{\zeta_i} ... \zeta_n$

Proposition 4.24. *The following super-bivector defines a Poisson bracket on $\mathcal{ST}^x(S_{(g,(P))})$:*

$$\pi_{SWP} = \sum_{\alpha,\beta \in E(\Gamma)} \epsilon^{\alpha\beta} x_\alpha x_\beta \frac{\partial}{\partial x_\alpha} \otimes \frac{\partial}{\partial x_\beta} - \frac{1}{2} \sum_{k \in F(\Gamma)} \overleftarrow{\partial_k}\overrightarrow{\partial_k}$$

This Poisson super-bracket does not depend on the peculiar triangulation.

Proof. It can be found in [Bou13], as well as other interesting facts, for example that flips and switches are super-Poisson maps. ☐

4.8 $\hat{\mathcal{ST}}^x(S_{(g,(P))})$ as $\mathcal{M}^{(G(\mathbb{R})_0^\times)_+}(B(\Gamma_S), K)$

To conclude this work, let's state the following result which is merely a consequence of the previous sections. Let $S_{(g,(P))}$ be any ciliated surface with a triangulation Γ. Consider the bipartite graph $B(\Gamma)$ associated to Γ. $H_1(B(\Gamma), \mathbb{Z}) \simeq \mathbb{Z}^{6g-6+3s+c}$ and the number of black vertices of $B(\Gamma)$ is $4g-4+2s+c$. Choose a labelling of the vertices of $B(\Gamma)$. Then any super structure on $S_{(g,(P))}$ corresponds to a matrix as in 4.5. Conversely, given a matrix which corresponds to a complex structure on $S_{(g,(P))}$ for some triangulation Γ and some Kasteleyn orientation K, any permutation of the rows or of the even columns corresponds to a relabelling of the vertices of $B(\Gamma)$. Hence:

Theorem 4.25.

$$\hat{\mathcal{ST}}^x(S_{(g,(P))}) \simeq \bigcap_{i=1}^{p} m_i^{-1}((G(\mathbb{R})_0^\times)_+)\mathcal{M}^{(G(\mathbb{R})_0^\times)_+}(B(\Gamma_S), K)_1$$

where by moduli space we mean moduli space of $(G(\mathbb{R})_0^\times)_+$-connections on $B(\Gamma_S)$ as well as the data of an odd number in $G(\mathbb{R})_1$ for each black vertex of $B(\Gamma_S)$, where i labels a basis of $H^1(B(\Gamma), (G(\mathbb{R})_0^\times)_+)$, and where the m_i are the monodromies around each loop i forming the basis of $H^1(B(\Gamma), (G(\mathbb{R})_0^\times)_+)$ as in the classical case.

Remark 30. *The parametrisation of this moduli space of course depends on Γ and on K. It emphasise the cluster nature of $\mathcal{ST}^x(S_{(g,(P))})$. To a choice of Γ and K there corresponds a global chart on $\mathcal{ST}^x(S_{(g,(P))})$. In section 4.5 we have computed the transition functions from one chart to another.*

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