Phases of $\mathcal{N} = 1$ SQCD and Seiberg Duality

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Preliminaries

General setup

SQCD is an $\mathcal{N} = 1$ supersymmetric gauge theory, with gauge group $\mathrm{SU}(N)$, quark (chiral) superfields Q and $\tilde{Q}$ that are respectively in the fundamental and anti-fundamental representation, coming in F flavors (hence Q and $\tilde{Q}$ are $F \times N$ matrices). The (UV) Lagrangian density of SQCD is:

$$\mathcal{L}_{\mathrm{SQCD}} = \frac{1}{32\pi} \mathrm{Im}\left\{\tau \int \mathrm{d}^2\theta W^\alpha W_\alpha\right\} + \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}((Q_i^a)^\dagger e^{2V_a^b} Q_b^i + (\tilde{Q}_i^a)^\dagger e^{-2V_a^b} \tilde{Q}_b^i) \tag{1}$$

Instantons, anomalies and holomorphy

Instantons Yang-Mills theories admit a Θ-term:

$$S_\Theta = \frac{\Theta}{32\pi^2} \int \mathrm{d}^4x \, \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \tag{2}$$

where $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. The Θ-term is a total derivative, hence this term has no effect on the classical equations of motions. However, the theta term plays a role in the quantized theory. A field configuration that is a classical solution of the (euclidean) action and approaches pure gauge at infinity is called an *instanton*. It is a fact that any simple Lie group G has $\pi_3(G) = \mathbb{Z}$, hence the Θ-term for such a G is an integer called the instanton number.

Instantons are intrinsically non-perturbative. This can be seen as follows. The running of the gauge coupling g with the scale μ reads:

$$\mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2} g^3 + \mathcal{O}(g^5) \ , \tag{3}$$

with b_1 an easy-to-compute constant depending on the structure of the theory. Hence, at one loop:

$$\frac{1}{g^2(\mu)} = -\frac{b_1}{8\pi^2} \log\left(\frac{\Lambda}{\mu}\right) \ , \tag{4}$$

where Λ is the scale at which the one-loop coupling diverges:

$$\Lambda = \mu \exp\left(-\frac{8\pi^2}{b_1 g^2(\mu)}\right) \ , \tag{5}$$

Remark 1. *Λ does not run, as one can easily check that* $\frac{\partial \Lambda}{\partial \mu} = 0$.

Now, one has:

$$0 \leq \int \mathrm{d}^4x \, \mathrm{Tr}\Big(F_{\mu\nu} \pm \tilde{F}^{\mu\nu}\Big)^2 = \int \mathrm{d}^4x [2 \, \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \pm 2 F_{\mu\nu} \tilde{F}^{\mu\nu}] \ . \tag{6}$$

Hence:

$$\int \mathrm{d}^4x \, \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \geq \left| \int \mathrm{d}^4x \, \mathrm{Tr}\, F_{\mu\nu} \tilde{F}^{\mu\nu} \right| = 32\pi^2 n \ , \tag{7}$$

where n is the instanton number. That last inequality implies that there is a lower bound for the instanton action, which is (multiply Eq. 7 by $1/4g^2$):

$$\mathrm{e}^{-S_{\rm inst}} = (\mathrm{e}^{\frac{8\pi^2}{g^2(\mu)}})^n = (\frac{\Lambda}{\mu})^{n b_1} \ , \tag{8}$$

showing what we wanted to show, namely that instantons are intrinsically non-perturbative.

Anomalies Anomalies are classical symmetries of the action that are broken by quantum effects. Local currents cannot be anomalous, since otherwise it wound break the unitarity of the theory. Hence *gauge currents cannot be anomalous*. However, global currents can be anomalous. In the sequel we will consider *chiral currents*, which arise in field theories in which fermions with chiral symmetries are coupled to gauge fields. In four-dimensions, anomalies only get contribution at one-loop through the *triangle diagrams* displayed in Fig. 1.

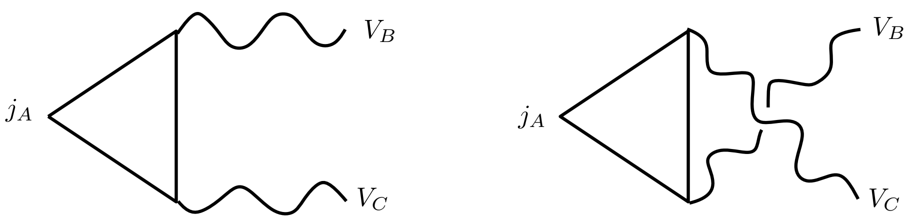


Figure 1: The two triangle diagrams contributing to the anomalies of j_A.

If j_A is a global current and j_B, j_C are two local currents appearing in the Lagrangian as $\mathcal{L} \ni V^B_\mu j^\mu_B + V^C_\mu j^\mu_C$, differentiating the correlator $\langle j_A V_B V_C \rangle$ yields:

$$\partial_\mu j^\mu_A \sim \mathrm{Tr}(t_A \{t_B, t_C\}) F^{\mu\nu}_B \tilde{F}_{C\mu\nu} \ . \tag{9}$$

From Eq. 9 one can note that:

- there is a link between instanton configurations and anomalies,

- only massless chiral fermions can contribute to the anomaly (otherwise the representation is real or pseudo-real, and $t_A = -(t_A)^T$, which implies $\mathrm{Tr}(t_A\{t_B, t_C\}) = -\mathrm{Tr}(t_A\{t_B, t_C\}) = 0$).

Now, consider a gauge theory with gauge group G (the generators of $\mathrm{Lie}(G) = \mathfrak{g}$ are denoted t_A) and global symmetry group $\tilde{G}$ (the generators of $\mathrm{Lie}(\tilde{G}) = \tilde{\mathfrak{g}}$ are denoted $\tilde{t}_A$). Suppose that there is a set of Weyl fermions label by i, such that the i-th one is in the representation $(r_i, \tilde{r}_i)$ of $G \times \tilde{G}$. Then the ABJ (standing for Adler-Bell-Jackiw) anomaly computation yields

$$\partial_\mu j^\mu_A \sim \sum_i \mathrm{Tr}_{\tilde{r}_i} \tilde{t}_A \, \mathrm{Tr}_{r_i}(t_b t_c) F^{\mu\nu}_B \tilde{F}_{C\mu\nu} \tag{10}$$

For any simple algebra $\tilde{\mathfrak{g}}$, it is a fact that $\mathrm{Tr}_{\tilde{r}_i}(\tilde{t}_A) = 0$. Hence only $U(1) \subset \tilde{G}$ can be anomalous. Then, $\mathrm{Tr}_{r_i}(t_B t_C) = C(r_i)\delta_{BC}$, where C is the quadratic Casimir of the representation r_i. In the end, for an abelian group one gets

$$\partial_\mu j^\mu_A = \frac{A}{16\pi^2} F^{\mu\nu}_B \tilde{F}_{B\mu\nu} \ , \tag{11}$$

where $A = \sum_i q_i C(r_i)$ is the *anomaly coefficient*, and where the q_i is the $U(1)$ charge of the i-th Weyl fermion.

Integrating Eq. 11 in space-time, one gets that the amount of charge violation due to the anomaly is $\Delta Q = 2An$, where n is the instanton number of the configuration.

Remark 2. *Anomalies in gauge theories are related to very deep pieces of mathematics, for which we refer to [Bil08] and [Har05], and also the Scholarpedia entry [Jac08].*

As an example (eyeing at what follows), consider the global symmetry group of $SQCD$ with gauge group $\mathrm{SU}(N)$ and F flavors. The charges of the elementary fields are given in Fig. 2.

	$\mathrm{SU}(F)_L$	$\mathrm{SU}(F)_R$	$U(1)_B$	$U(1)_A$	$U(1)_{R_0}$
Q^i_a	F	0	1	1	a
$\tilde{Q}^b_j$	0	$\bar{F}$	-1	1	a
λ	0	0	0	0	1

Figure 2: Charges of the fields in SQCD under the global symmetry group.

One easily computes $A_A = F$ and $A_{R_0} = N + (a-1)F$, hence the axial and R_0 symmetries are anomalous. However, one can define

$$j^R_\mu = j^{R_0}_\mu + \frac{(1-a)F - N}{F} j^A_\mu \tag{12}$$

which is not anomalous. Hence the quantum symmetries of SQCD are $G_F = \mathrm{SU}(F)_L \times \mathrm{SU}(F)_R \times U(1)_B \times U(1)_R$.

Remark 3. *When $F = 0$ (pure SYM), there is no axial current, hence the R symmetry is anomalous.*

't Hooft anomaly matching condition In short, 't Hooft anomaly matching condition arises in the following picture. Consider a theory with global symmetry group G. Let A_{UV} be the anomaly coefficient of this G. Then gauge the group G (with gauge coupling g), and add fermions to cancel the anomaly, i.e., so that they contribute $A_s = -A_{UV}$. Follow the renormalisation group flow to the IR, and simultaneously consider the limit $g \to 0$. In that limit the fermions decouple, however the theory is still anomaly free, hence $A_{IR} = -A_s = A_{UV}$.

Holomorphy is the coupling constants, and the requirement of smoothness of physics in various weak-coupling limits places constraints of the effective super-potential.

Non-renormalisation of the beta function Recall that the beta function of SQCD with gauge group $\mathrm{SU}(N)$ and F flavors is

$$\beta = \mu \frac{\partial g}{\partial \mu} = -\frac{b_1}{16\pi^2} g^3 + \mathcal{O}(g^5) \ , \tag{13}$$

with $b_1(\mathrm{SQCD}) = 3N - F$. Can holomorphy tell us something about higher-loop (and non-perturbative) corrections to this one-loop β?

Consider SYM theory to make things simpler, the case of SQCD being completely similar. The lagrangian density reads:

$$\mathcal{L} = \frac{1}{16\pi i} \int \mathrm{d}^2\theta \tau \, \mathrm{Tr}\, W^\alpha W_\alpha + h.c. \tag{14}$$

where $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$ is the *complexified gauge coupling*. The running given by the *beta* function is

$$\frac{1}{g^2(\mu)} = -\frac{3N}{8\pi^2} \log\left(\frac{|\Lambda|}{\mu}\right) \ , \tag{15}$$

hence $|\Lambda| = \mu_0 \exp\Big(-\frac{8\pi^2}{3Ng^2(\mu_0)}\Big)$. One defines a *holomorphic scale*

$$\Lambda = |\Lambda| e^{i\frac{\Theta}{3N}} = \mu e^{\frac{2\pi i \tau}{3N}} \ . \tag{16}$$

Equivalently, $\tau_{1-\text{loop}} = \frac{2N}{2\pi i}\log\frac{\Lambda}{\mu}$. Now, note that physics is periodic under $\Theta \to \Theta + 2\pi$, hence $\Lambda \to \exp\big(\frac{2\pi i}{3N}\big)\Lambda$ is a symmetry, and so is $\tau \to \tau + 1$. The most general $\tau(\Lambda, \mu)$ that respects the latter equation is of the form

$$\tau(\Lambda,\mu) = \frac{3N}{2\pi i}\log\left(\frac{\Lambda}{\mu}\right) + f(\Lambda,\mu) \ , \tag{17}$$

where f is a holomorphic function of Λ, such that it has a positive Taylor expansion is Λ is order to yield to classical result as $\Lambda \to 0$, and such that it is invariant under $\Lambda \to \exp\big(\frac{2\pi i}{3N}\big)\Lambda$. Hence:

$$\tau(\Lambda,\mu) = \frac{3N}{2\pi i}\log(\Lambda\mu) + \sum_{n=1}^{\infty} a_n \Big(\frac{\Lambda}{\mu}\Big)^{3Nn} \ . \tag{18}$$

Hence we proved that the beta function in SYM is actually one-loop exact. The reasoning above transposes directly to the SQCD case, with the replacement $b_1 = 3N - F$ instead of $b_1 = 3N$.

Remark 4. *In some cases, one can also show that non-perturbative corrections are absent. One should also pay attention to the difference between holomorphic normalisation and physical normalisation of fields. In turns out that the transformation from one set of fields to the other is singular, and explains the apparent discrepancy between the one-loop exact beta function presented above and the NSVZ beta function (Novikov-Shifman-Vainshtein-Zakharov), which reads*

$$\beta = -\frac{g^3}{16\pi^2}\frac{[3N - \sum_{i=1}^{F}(1-\gamma_i)]}{1 - \frac{Ng^2}{8\pi^2}} \ . \tag{19}$$

The $\gamma_i = d\log Z_i(\mu)/d\log(\mu)$ are the anomalous dimensions of the matter fields. It is clear that the NSVZ beta functions gets contributions at all loops. See pages 188-191 of [Ber19] for more details on that matter.

Perturbative analysis of $\mathcal{N} = 1$ SQCD

We know that $\mathcal{N} = 1$ SQCD is a renormalizable supersymmetric gauge theory, with gauge group $\mathrm{SU}(N)$, F flavors $(Q, \tilde{Q})$ and no superpotential. There are interaction terms, but they come from the D-terms only. These read

$$D^A = (Q_i^b)^\dagger (T^A)_b^c Q_c^i - \tilde{Q}_i^b (T^A)_b^c (\tilde{Q}_c^i)^\dagger \tag{20}$$

where $A = 1, 2, ..., N^2 - 1$ label the generators of adjoint representation of $\mathrm{SU}(N)$, where $a, b, ...$ label gauge indices while $i, j, ...$ label color indices.

Semi-classically, there are two different cases:

- $\underline{F < N}$ The quark matrices Q and $\tilde{Q}$ can be put under the form

$$Q = \begin{bmatrix} v_1 & & & \\ & \ddots & & (0) \\ & & v_F & \end{bmatrix} = \tilde{Q}^T \ . \tag{21}$$

  At a generic point of the (classical) moduli space, the gauge group $\mathrm{SU}(N)$ is broken to $\mathrm{SU}(N-F)$. The *gauge invariant* mesonic operators $M^i_j = Q^i_a \tilde{Q}^a_j$ parametrize the classical moduli space $\mathcal{M}_{\text{cl}}$. For $F < N$, the meson matrix has maximal rank, hence there is no classical constraint. The dimension of $\mathcal{M}_{\text{cl}}$ is

$$\dim_{\mathbb{C}}(\mathcal{M}_{\text{cl}}) = 2FN - \{N^2 - 1 - [(N-F)^2 - 1]\} = F^2 \ . \tag{22}$$

- $\underline{F \geq N}$ The quark matrices Q and $\tilde{Q}$ can be put under the form

$$Q = \begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_F \\ & (0) & \end{bmatrix} \ , \tag{23}$$

  and

$$\tilde{Q}^T = \begin{bmatrix} \tilde{v}_1 & & \\ & \ddots & \\ & & \tilde{v}_F \\ & (0) & \end{bmatrix} \ , \tag{24}$$

on which the D-term equations impose $|v_i|^2 - |\tilde{v}_i|^2 = a$, with a a constant that does not depend on i. At a generic point of $\mathcal{M}_{\rm cl}$, the gauge group is now fully broken, and:

$$\dim_{\mathbb{C}}(\mathcal{M}_{\rm cl}) = 2FN - (N^2-1) \ . \tag{25}$$

The *gauge invariant operators* are now the mesonic ones $M^i_j = Q^i_a \tilde{Q}^a_j$, together with the baryons:

$$B_{i_1...i_{F-N}} = \epsilon_{i_1...i_{F-N}j_1...j_N}\epsilon^{a_1...a_N} Q^{j_1}_{a_1}...Q^{j_N}_{a_N} \ , \tag{26}$$

and the anti-baryons:

$$\tilde{B}^{i_1...i_{F-N}} = \epsilon^{i_1...i_{F-N}j_1...j_N}\epsilon_{a_1...a_N} \tilde{Q}^{a_1}_{j_1}...\tilde{Q}^{a_N}_{j_N} \ . \tag{27}$$

There is one classical constraint between these variables, coming from the very definition of M, which is:

$$\det M - B\tilde{B} = 0 \tag{28}$$

1 Pure SYM: gaugino condensation

Recall first that pure SYM is the case where there is no anomaly-free R-symmetry:

$$\partial_\mu j^\mu_R = 0 \to \partial_\mu j^\mu_R = 2N \frac{1}{32\pi^2} F^a_{\mu\nu}\tilde{F}^{\mu\nu}_a \ , \tag{29}$$

since $A_{R_0} = \sum q_i C(r_i) = N$, where these notations refer to what follows Fig. 2. The $U(1)_R$ global symmetry group gets spontaneously broken to $\mathbb{Z}_{2N}$. Using holomorphicity arguments, one can easily get the structure of the effective superpotential $W_{\rm eff}$, if any. First notice that the operator $e^{2\pi i\tau/N}$ has R-charge 2, since the theta angle transforms as $\Theta \to \Theta + 2N\alpha$ under R-symmetry transformations.

Because of confinement, and *under the assumption of a mass gap*, the effective Lagrangian should only depend on τ, hence $W_{\rm eff}$ should also depend only on τ. Imposing $\mathbb{Z}_{2N}$ R-symmetry, by dimensional analysis, the only possible term is

$$W_{\rm eff} = c\mu^3 e^{2\pi i\tau/N} = c\Lambda^3 \tag{30}$$

where c is a constant to be determined (possibly 0). Recall the (UV) Lagrangian of SYM:

$$\mathcal{L} = \frac{1}{32\pi}\,{\rm Im}\left[\int {\rm d}^2\theta\tau\,{\rm Tr}\,W^\alpha W_\alpha\right] \ . \tag{31}$$

The scalar component of $W^\alpha W_\alpha$ is the gaugino bilinear $\lambda^\alpha\lambda_\alpha$. We think τ as a (spurious) chiral superfield: $\tau = \tau + \sqrt{2}\theta\psi_\tau - \theta\theta F_\tau$, and as such, $-F_\tau$ acts as a source for $\lambda^\alpha\lambda_\alpha$. Then, writing $Z = \int {\rm D}V e^{i\int\mathcal{L}}$:

$$\langle\lambda\lambda\rangle = 16\pi i \frac{\partial}{\partial F_\tau}\log Z = 16\pi i\frac{\partial}{\partial\tau}W_{\rm eff}(\tau) \ , \tag{32}$$

and using Eq. 30:

$$\langle\lambda\lambda\rangle = a\Lambda^3 \ , \tag{33}$$

with $a = \frac{-32\pi^2}{N}c$. Hence, *if* $c \neq 0$, gauginos condensate and since the R-charge of the gauginos bilinears is 1, the discrete $\mathbb{Z}_{2N}$ R-symmetry group get further broken to $\mathbb{Z}_2$ by the choice of a vacuum. There are N vacua since $\Theta \to \Theta + 2\pi k$ is a symmetry of the theory, and acts as $\langle\lambda\lambda\rangle \to e^{2i\alpha}\langle\lambda\lambda\rangle$.

2 SQCD for $0 < F < N$: ADS superpotential

The perturbative analysis of this case (see Eq. 22) yields a classical moduli space of complex dimension F^2n parametrized by mesons field VEVs. One question we would like to answer is: as one flows to the IR, is there a non-zero effective super-potential ? To answer that, we promote the complexified coupling and the dynamical scale to spurion fields. At low energy, an effective superpotential can only depend on the meson matrix M, and on the complexified gauge coupling, through Λ. The quantum numbers of these fields are:

	$U(1)_B$	$U(1)_A$	$U(1)_R$
$\det M$	0	$2F$	$2(F-N)$
Λ^{3N-F}	0	$2F$	0

(note that $\det M$ is the only invariant under $\mathrm{SU}(F)_L\times\mathrm{SU}(F)_R$ one can make out of M) hence the only superpotential that can be generated has the form:

$$W_{\rm eff} = c_{N,F}\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \tag{34}$$

Now one can wonder, what the properties of this superpotential are, what its physical origin is, and if one can compute the coefficient c_{N-F}, what its value is.

2.1 Scale matching

At a generic point of the moduli space, the gauge group SU(N) is spontaneously broken to SU($N-F$), and the meson matrix takes the form given in Eq. 21. Suppose for simplicity that $v_i = v$, for all i, and some v. At energies above v, the gauge coupling running is that of SQCD with gauge group SU(N) and F flavors:

$$\frac{4\pi}{g^2(\mu)} = \frac{3N-F}{2\pi} \log \frac{\Lambda}{\mu} \ . \tag{35}$$

On the other hand, for $E < v$ the running is that of SU($N-F$) SYM:

$$\frac{4\pi}{g_L^2(\mu)} = \frac{3(N-F)}{2\pi} \log \frac{\Lambda_L}{\mu} \ , \tag{36}$$

where we have added a subscript L to the gauge coupling and the dynamically generated scale which a priori, are different. If SUSY is preserved, then:

$$g(v) = g_L(v) \ . \tag{37}$$

Eq. 37 is known as the *scale matching* condition. From that one gets:

$$\Lambda_L^{3(N-F)} = \frac{\Lambda^{3N-F}}{v^{2F}} = \frac{\Lambda^{3N-F}}{\det M} \ , \tag{38}$$

hence that:

$$\Lambda_L^3 = \left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \ . \tag{39}$$

From this we see that $W_{\text{eff}} = c_{N,F}\Lambda_L^3$, hence that $c_{N,F} = c_{N-F,0}$, and also that W_{eff} is generated by gaugino condensation of the left-over SU($N-F$) gauge group.

Holomorphic decoupling can also tell us some information about these $c_{N,F}$. Consider SQCD with N colors and F flavors, and add a mass m to the F-th flavor. For $E > m$, the running is that of SQCD with N colors and F flavors, however, when $E < m$, the F-th flavor decouples and one is effectively left with SQCD with N colors and $F-1$ flavors. The scale matching condition $g(m) = g_L(m)$ gives $\Lambda_{L,F-1}^{3N-F-1} = m\Lambda_F^{3N-F}$, hence:

$$W_{\text{eff}} = c_{N,F-1}\left(\frac{\Lambda_{L,F-1}^{3N-F+1}}{\det \tilde{M}}\right)^{\frac{1}{N-F+1}} = c_{N,F-1}\left(\frac{m\Lambda_F^{3N-F}}{\det \tilde{M}}\right)^{\frac{1}{N-F+1}} \tag{40}$$

which is equivalent to:

$$c_{N,F-1} = (N-F+1)\left(\frac{c_{N,F}}{N-F}\right)^{\frac{N-F}{N-F+1}} \tag{41}$$

Eventually, using the fact that $c_{N,F} = c_{N-F,0}$, we obtain

$$c_{N,F} = (N-F)c^{\frac{1}{N-F}} \ , \tag{42}$$

with c a constant to be determined (that could, for now, very well be zero). Consider the special case $F = N-1$, for which $c_{N,N-1} = c$. The gauge group in this case is generically fully broken, hence there is no left-over strong IR dynamics, hence every term appearing in the effective action should be visible in a weak-coupling analysis. Moreover, $W_{\text{eff}} = \Lambda^{2N+1}$, which is the one-instanton contribution to the amplitude (in this case $b_1 = 2N+1$), suggesting that this effective superpotential is generated by instanton effects in this case. At weak coupling, an instanton calculation can be done and it shows that $c = 1$, hence that $c_{N,F} = N-F$ for all N, and $F < N$.

To sum up, we found that in the IR, an effective superpotential is generated by gaugino condensation of the unbroken subgroup SU($N-F$). It reads:

$$W_{\text{ADS}} = (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{\frac{1}{N-F}} \ , \tag{43}$$

and it is called the ADS effective superpotential (standing for Affleck-Dine-Seiberg). Note that the corresponding potential is runaway:

$$V_{\text{ADS}} = \sum_i \left|\frac{\partial W_{\text{ADS}}}{\partial Q_i}\right|^2 + \left|\frac{\partial W_{\text{ADS}}}{\partial \tilde{Q}_i}\right|^2 \tag{44}$$

is minimized for $Q, \tilde{Q} \to \infty$.

2.2 The linearity principle

Sometimes the potential of SU(N) SYM theory is written as:

$$W_{\mathrm{VY}} = NS(1 - \log \frac{S}{\Lambda^3}) \ , \tag{45}$$

where $S = -\frac{1}{32\pi^2}\,\mathrm{Tr}(W^\alpha W_\alpha)$ is called the *glueball superfield*. The potential in Eq. is known as the Veneziano-Yankielowicz superpotential. One observes that integrating S out (since we are *supposing* that SYM theories have a *mass gap*) yields:

$$\frac{\partial W_{\mathrm{VY}}}{\partial S} = N(1 - \log \frac{1}{\Lambda^3}) + NS(-\frac{1}{S}) = 0, \text{ hence } \langle S \rangle = \Lambda^3 \ . \tag{46}$$

Replacing S W_{VY} by Λ^3 yields the effective SU(N) SYM superpotential $W_{\mathrm{eff}} = N\Lambda^3$. Conversely, one can integrate S *in* in the latter to retrieve the Veneziano-Yankielowicz superpotential.

In the case of SQCD, the situation is pretty much similar: starting from the ADS potential given in Eq. 43 one can integrate S *in* to yield the Taylor-Veneziano-Yankielowicz superpotential:

$$W_{\mathrm{TVY}} = (N-F)S\left[1 - \frac{1}{N-F}\log\left(\frac{S^{N-F}\det M}{\Lambda^{3N-F}}\right)\right] \ . \tag{47}$$

Conversely, starting from W_{TVY} and integrating S out yields W_{ADS}.

The general case Consider a general $\mathcal{N}=1$ gauge theory with a tree-level superpotential of the form

$$W_{\mathrm{UV}} = \sum_r \lambda_r X_r(\Phi_i) \ , \tag{48}$$

where X_r are gauge-invariant operators constructed out of the chiral superfields Φ_i, and where the λ_r are the corresponding couplings. One can expect the non-perturbative superpotential to depend holomorphically on λ_r, X_r and Λ_s (the strong coupling scales). Actually, Intrilligator, Leigh and Seiberg have shown that W_{NP} does not depend on the couplings λ_r, hence in general the effective superpotential

$$W_{\mathrm{eff}} = \sum_r \lambda_r X_r + W_{\mathrm{NP}}(X_r, \Lambda_s) \tag{49}$$

is *linear in the couplings*. Hence, at low enough energy where the superpotential terms dominate with respect to the kinetic terms, one can integrate out the X_r using their F-term equation, which reads $\lambda_r = -\frac{\partial}{\partial X_r} W_{NP}$. Each pair (λ_r, X_r) behaves as a pair of Legendre dual variables! Solving X_r in terms of the couplings and the dynamical scales yields an effective superpotential that only depend on the latter:

$$W_{\mathrm{eff}}(\lambda_r, \Lambda_s) = \left[\sum_r \lambda_r X_r + W_{\mathrm{NP}}(X_r, \Lambda_s)\right]_{X_r(\lambda_r, \Lambda_s)} \ . \tag{50}$$

However, since the Legendre transform is invertible, one can reverse the procedure and integrate the X_r in, as:

$$\langle X_r \rangle = -\frac{\partial}{\partial \lambda_r} W_{eff}(\lambda_r, \Lambda_s) \ . \tag{51}$$

Note that these descriptions are equivalent only because we are not considering D-terms, hence this equivalence is more and more correct, the lower the energy.

Dual dynamical scales and glueballs One can rewrite the gauge kinetic term as a superpotential term:

$$W_{\mathrm{tree}} = \frac{\tau(\mu)}{16\pi i}\,\mathrm{Tr}\, W^\alpha W_\alpha = 3N \log\left(\frac{\Lambda}{\mu}\right) S \ , \tag{52}$$

hence one can think to Λ and S as Legendre dual variables. This point of view allows to go from the SYM superpotential to the VY one, and from the ADS superpotential to the TVY one.

3 SQCD for $N = F$ and $F = N + 1$

Upshot A properly defined effective superpotential cannot be generated, as there is no way of constructing an object respecting all the symmetries, with the correction mass dimension, vanishing in the classical limit and with R-charge 2, using the coupling and fields we have - $M, B, \tilde{B}$ and Λ. However, and even if the classical moduli space is not lifted by strong dynamical effects, it is *deformed* by the latter.

3.1 $F = N$

All gauge invariant operators have R charge $R = 0$, hence one cannot construct an effective superpotential with R-charge 2. The gauge invariant operators are the mesons, and two baryons $B = \epsilon^{a_1 \dots a_N} Q^1_{a_1} \dots Q^N_{a_N}$ and $\tilde{B} = \epsilon_{a_1 \dots a_N} \tilde{Q}^{a_1}_1 \dots \tilde{Q}^{a_N}_N$. The classical moduli space has complex dimension $N^2 + 1$, and is parametrized by the (singular) hypersurface in $\mathbb{C}^{N^2+2}$ (parametrized by the entries of M, B and $\tilde{B}$) corresponding to the equation $\det M - B\tilde{B} = 0$.

The quantum deformation At a quantum level, one could expect the classical constraint to be modified as

$$\det M - B\tilde{B} = a\Lambda^{2N} \ . \tag{53}$$

This deformation is allowed, in terms of symmetries, and as is had to be the case, in the classical limit one retrieves the classical constraint since then, $\Lambda \to 0$. The exponent $2N$ is the one-loop coefficient of the β function, hence it is tempting to think to this deformation as a one-instanton correction. Let's implement Eq. 53 in the superpotential with a Lagrange multiplier:

$$W = A(\det M - B\tilde{B} - a\Lambda^{2N}) \ . \tag{54}$$

Using holomorphic decoupling, i.e. adding a mass term to the F-th flavor, at low energy one gets SQCD with $F = N - 1$, and to get the ADS superpotential out of Eq. 54, one needs $a = 1$. Hence:

$$\det M - B\tilde{B} = \Lambda^{2N} \ . \tag{55}$$

Remark 5. *The quantum moduli space* $\mathcal{M}_{\text{qu}}$ *is* smooth. *The origin is excised of it, hence* any allow vacuum breaks chiral symmetry.

At a generic point of $\mathcal{M}_{\text{qu}}$, where all the GIO get a VEV, all global symmetries are broken. However, there are submanifolds of enhanced global symmetry:

- The *mesonic branch* is defined by $B = \tilde{B} = 0$, and $M^j_i = \Lambda^2 \delta^j_i$. On it, the global UV symmetry group $G = \text{SU}(F)_L \times \text{SU}(F)_R \times U(1)_B \times U(1)_R$ is broken to $\text{SU}(F)_D \times U(1)_B \times U(1)_R$ for a diagonal $\text{SU}(F)_D$.
- The *baryonic branch* is defined by $M = 0$, $B = -\tilde{B} = \Lambda^N$. There, G gets broken to $\text{SU}(F)_L \times \text{SU}(F)_R \times U(1)_R$.

The phases of these theories are summarized in the following Fig. 3.

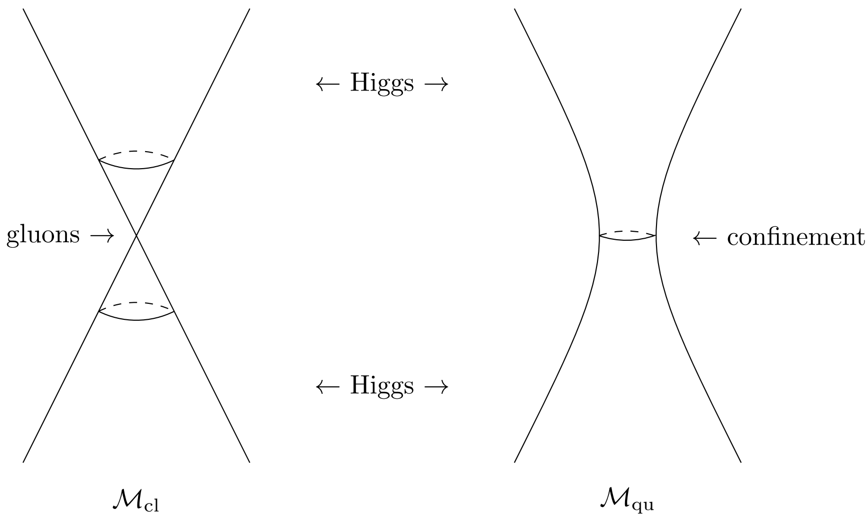


Figure 3: The phases of SQCD for $F = N$.

Consistency checks One can compute 't Hooft anomalies in the UV and in the IR, e.g. on the mesonic branch. The global symmetry is $\text{SU}(F)_D \times U(1)_B \times U(1)_R$, and the charges of the different fields read:

	$\text{SU}(F)_D$	$U(1)_B$	$U(1)_R$
ψ_Q	F	1	-1
$\psi_{\tilde{Q}}$	$\bar{F}$	-1	-1
λ	0	0	1
ψ_M	Adj	0	-1
ψ_B	0	F	-1
λ	0	$-F$	-1

where the upper-half (resp. lower-part) of this table concerns the UV (resp. IR) degrees of freedom. Note that we have used the quantum relation Eq. 55 in order to eliminate the fermionic partner of Tr M. The anomaly calculation yields:

	UV	IR
$\mathrm{SU}(F)_D^2 \times U(1)_R$	$2N(-1)/2 = -N$	$F(-1) = -F$
$U(1)_B^2 \times U(1)_R$	$-2NF$	$-2F^2$
$U(1)_R^3$	$-2NF + N^2 - 1$	$-(F^2-1) - 1 - 1 = -F^2 - 1$

Since $F = N$, 't Hooft anomaly matching conditions hold. This constitutes a non-trivial check of the proposal. One can do the same on the mesonic branch, and find as well that 't Hooft AMC are satisfied.

3.2 $F = N + 1$

For $F = N + 1$, the classical moduli space is parametrised by mesons and baryons, however, now the baryons carry one flavor index:

$$B_i = \epsilon_{ij_1\ldots j_n}\epsilon^{a_1\ldots a_N}Q_{a_1}^{j_1}\ldots Q_{a_n}^{j_n} \ , \tag{56}$$

and similarly for $\tilde{B}^i$. The classical moduli space is actually quantum exact, as one can prove by holomorphic decoupling. Seiberg showed that the only effective superpotential respecting the symmetries, having correct R-charge and correct dimension, necessarily is of the form

$$W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}}(\det M - B_i M_j^i \tilde{B}^j) \ , \tag{57}$$

where a is a constant that needs to be fixed. Note that since the rank of M is $k < N$, classically one has $\det M = 0$, hence 57 is to be thought of as a quantum equation (valid off-shell). By adding a mass term to the F-th flavor, one can see that $a = 1$ in order to recover the results of the previous sections. This roughly goes as follows:

$$W_{\text{eff}} = \frac{a}{\Lambda^{2N-1}}(\det M - B_i M_j^i \tilde{B}^j) - m M_{N+1}^{N+1} \tag{58}$$

The F-flatness equations for M_i^{N+1}, M_{N+1}^i, B_i and $\tilde{B}^i$ implies that

$$M = \begin{pmatrix} \hat{M}_{ij} & 0 \\ 0 & t \end{pmatrix}, \ B = \begin{pmatrix} 0_i \\ \hat{B} \end{pmatrix}, \ \tilde{B} = \begin{pmatrix} 0_i \\ \hat{\tilde{B}} \end{pmatrix}, \tag{59}$$

where $t = M_{N+1,N+1}$. Then, the F-flatness equation for t implies

$$\frac{a}{\Lambda^{2N-1}}(\det \hat{M} - \hat{B}\hat{\tilde{B}}) - m = 0 \ \Rightarrow \det \hat{M} - \hat{B}\hat{\tilde{B}} = \frac{1}{a}\Lambda_L^{2N} \ , \tag{60}$$

hence $a = 1$. By differentiating Eq. 57, one gets the equations parametrizing the moduli space:

$$\begin{cases} M \cdot \tilde{B} = B \cdot M = 0 \\ \det M \cdot (M^{-1})_i^j - B_i \tilde{B}^j = 0 \end{cases} . \tag{61}$$

Note that $\det M \cdot (M^{-1})_i^j$ is the i-th j-th minor of the matrix M, that is, $(-1)^{i+j}$ times the determinant of the matrix obtained from M by removing the i-th row and the j-th column.

Remark 6. *This technique does not apply to $F \geq N + 2$. It is not possible to built an effective superpotential with R-charge equals to 2, correct dimensions, and symmetries.*

Vacuum structure of SQCD with $F = N + 1$

- Conversely to the case $F = N$, here the point $M = B = \tilde{B} = 0$ belongs to $\mathcal{M}_{\text{qu}}$. At this point there is (confinement) charge screening without chiral symmetry breaking. Theories with such a property are said to be *s-confining.*
- As for $F = N$ SQCD, this theory exhibits *complementarity*: one can move smoothly from a confining phase (near 0) to a Higgs phase without any order parameter differentiating the two.

4 Conformal window

Seiberg did the following proposal, that SQCD with $\frac{3}{2}N < F < 3N$ flows to an interacting IR fixed point (hence does not confine). Recall from Eq. 19 that the NSVZ beta function of $\mathcal{N}=1$ SQCD with N colors and F flavors is:

$$\beta^{N,F}_{\mathrm{SQCD}} = \frac{-g^3}{16\pi^2}.\frac{3N - F[1-\gamma(g^2)]}{1 - Ng^2/8\pi^2} \ . \tag{62}$$

It is the beta function of the physical gauge couplings, and:

$$\gamma(g^2) = -\frac{g^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g^4) \tag{63}$$

is the anomalous dimension of matter fields. Hence:

$$\beta^{N,F}_{\mathrm{SQCD}} = \frac{-g^3}{16\pi^2}[3N - F + (3N^2 - 2NF + \frac{F}{N})\frac{g^2}{8\pi^2} + \mathcal{O}(g^4)] \ . \tag{64}$$

Assume now that F is slightly smaller than $3N$, i.e. $\epsilon = 3 - \frac{F}{N} << 1$. Then one can write:

$$\beta^{N,F}_{\mathrm{SQCD}} = \frac{-g^3}{16\pi^2}[\epsilon N - (3(N^2-1) + \mathcal{O}(\epsilon))\frac{g^2}{8\pi^2} + \mathcal{O}(g^4)] \ , \tag{65}$$

and this function has a zero at

$$g_*^2 = \frac{8\pi^2 N}{3(N^2-1)\epsilon} \ , \tag{66}$$

up to ϵ^2. It is called the Banks-Zaks fixed point. Seiberg argued that such a fixed point exists for

$$\frac{3}{2}N < F < 3N \ . \tag{67}$$

This interval is called the conformal window of SQCD with N colors and F flavors.

Lower bound of the conformal window. In a SCFT, the scaling dimension of an operator satisfies

$$\Delta \geq \frac{3}{2}|R| \tag{68}$$

and the equality holds for chiral and anti-chiral operators. For the meson matrix this implies

$$\Delta(M) = \frac{3}{2}R(M) = \frac{3}{2}R(Q\tilde{Q}) = 3\frac{F-N}{F} = 2 + \gamma_* \ , \tag{69}$$

since the anomalous dimension of the meson matrix at the fixed point is $\gamma_* = 1 - \frac{3N}{F}$. Besides, it has to be that $\Delta \geq 1$ (otherwise there is a negative norm state in the theory), hence $F = \frac{3N}{2}$ is a lower bound (there, $\Delta = 1$).

Upper bound of the conformal window. The condition $F \geq 3N$ is obviously an upper bound for the conformal window, since SQCD is in an IR free phase above that, with potential between probe charge $V \sim \frac{g^2}{r}$, and $g^2 \sim \frac{1}{\log(r\Lambda)}$.

5 Seiberg duality

5.1 The idea

Seiberg proposal is that the IR physics of SQCD for $F > N+1$ has an equivalent description in terms of another supersymmetric gauge theory, known as the *magnetic dual theory.*

Consider SQCD with N colors, and $F > N$. The gauge invariant operators in this theory are the mesons M_i^j, together with the baryons $B_{i_1 i_2 \ldots i_{F-N}}$ and the anti-baryons $\tilde{B}^{i_1 i_2 \ldots i_{F-N}}$. Baryons and anti-baryons indeed have $F-N$ free indices in that case, hence one might like to view the baryons as bound states of $\tilde{N}$ components (some new quark fields q and $\tilde{q}$ of some SYM with gauge group $\mathrm{SU}(\tilde{N}) = \mathrm{SU}(F-N)$), for which q and $\tilde{q}$ transform in the $\tilde{N}$ and $\bar{\tilde{N}}$ representations, respectively. Originally the baryon field is defined as

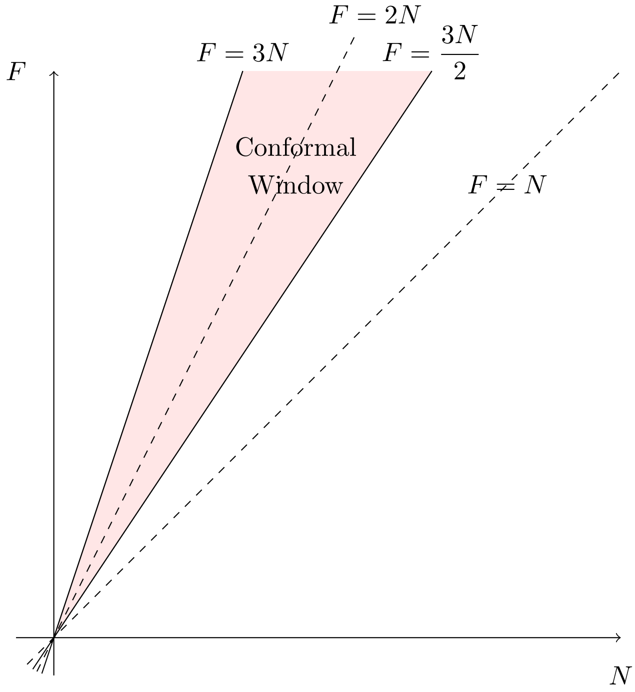
$$B_{i_1 i_2 \ldots i_{\tilde{N}}} = \epsilon_{i_1 \ldots i_{\tilde{N}} j_1 \ldots j_N} \epsilon^{a_1 \ldots a_N} Q^{j_1}_{a_1} \ldots Q^{j_N}_{a_N} \ , \tag{70}$$

and one wants to write it as:

$$B_{i_1 i_2 \ldots i_{\tilde{N}}} = \epsilon_{a_1 \ldots a_{\tilde{N}}} q^{a_1}_{i_1} \ldots q^{a_{\tilde{N}}}_{i_{\tilde{N}}} \ , \tag{71}$$

and similarly for the anti-baryons $\tilde{B}^{i_1 i_2 \ldots i_{F-N}}$. This naive physical wish can be given a rigorous sense, and the correspondence states that SQCD with gauge group $\mathrm{SU}(N)$ and F flavors is dual to a theory called *mSQCD* which is basically SQCD with gauge group $\mathrm{SU}(F-N)$ and F flavors, together with an additional chiral field Φ which is gauge-neutral but transforms in the $(\square_{F_L}, \bar{\square}_{F_R})$, and a superpotential term $W = q_i \Phi^i_j \tilde{q}^j$. The meaning of duality here is that these two theories have the same IR physics.

The conformal window. Let's first focus on the superconformal window (see Eq. 67). Since we assumed that $W = 0$, the field Φ is completely decoupled, and mSQCD is simply SQCD with gauge group $SU(F-N)$ and F flavors. Note the cool fact that the duality preserves the conformal windows: the conformal window of the original theory is mapped to the conformal window of the second.



Hence mSQCD without superpotential W flows to an IR fixed point for $\frac{3N}{2} < F < 3N$. At such a fixed point, $\Delta(W) = \Delta(\Phi)+\Delta(q)+\Delta(\tilde{q}) = 1+\frac{3N}{2F}+\frac{3N}{2F} < F$, hence W is a *relevant operator*. The claim is that this perturbation drives the theory to a new fixed point which turns out to be the same fixed point than the one of SQCD.

Below the window. The one loop coefficient of the beta function of *mSQCD* is $b_1 = 2F - 3N$, hence for $2F = 3N$, the beta function vanishes and for lower F, it changes sign, and mSQCD becomes IR free. Hence for $N + 1 < F \leq \frac{3N}{2}$, the dynamics of SQCD is best described by a theory of freely interacting combinations of mesons and baryon fields. This is called the *free magnetic phase* of SQCD. Since mSQCD is IR-free, in terms of magnetic dual variables the Kähler potential is canonical (up to $\frac{1}{\Lambda^2}$-corrections), hence we know the full effective IR Lagrangian for $N + 1 < F \leq \frac{3N}{2}$, at low enough energies!

Above the window. For $F \geq 3N$, the magnetic theory does not reach an IR interacting fixed point anymore. There, SQCD becomes IR free, and the dual mSQCD becomes asymptotically free.

5.2 Consistency checks

Global symmetries and IR degrees of freedom. At the IR fixed point, we expect the following correspondence between fields in SQCD and fields in mSQCD:

$$\begin{cases} M & \leftrightarrow & \Phi : \Phi^i_j = \frac{1}{\mu} M^i_j \\ B & \leftrightarrow & b = b^{j_1 \ldots j_N} = \epsilon^{i_1 \ldots i_{F-N} j_1 \ldots j_N} B_{i_1 \ldots i_{F-N}} \\ \tilde{B} & \leftrightarrow & \tilde{b} \end{cases} . \tag{72}$$

The scale μ is needed in order to match the scaling dimensions of the IR degrees of freedom. In SQCD, mesons are composite, and $\Delta(M) = 2$ in the UV (where SQCD is free). In mSQCD, Φ is elementary, and $\Delta(M) = 1$ in the UV. Under the RG flow, both M and Φ acquire anomalous dimensions, and should flow to the same operator in the IR, hence the need for the scale μ.

From the table in Fig. 4, one sees that the superpotential term has R-charge $R(W) = 2$, as expected. Moreover, one sees that SQCD and mSQCD have the same IR degrees of freedom: there is a one-to-one correspondence between gauge invariant operators, where the meson matrix of SQCD is mapped to Φ, and the baryons are mapped to the dual magnetic quarks. However, there does not seem to be any GIO is the SQCD theory, that is mapped to the meson matrix $U^j_i = q_i \tilde{q}^j$ of mSQCD. This is the very reason why one needs the superpotential W, since the F-term corresponding to Φ reads $F_\Phi = q\tilde{q} = U = 0$. Hence U vanishes identically on the moduli space.

	$\mathrm{SU}(F)_L$	$\mathrm{SU}(F)_R$	$U(1)_B$	$U(1)_R$
q_i^a	$\bar{F}$	0	$\frac{N}{F-N}$	$\frac{N}{F}$
$\tilde{q}_b^j$	0	F	$-\frac{N}{F-N}$	$\frac{N}{F}$
Φ	F	$\bar{F}$	0	$2\frac{F-N}{F}$
λ	0	0	0	1

Figure 4: Global symmetries of the UV degrees of freedom of mSQCD.

't Hooft anomaly matching condition. They hold, see p.238 in [Ber19].

Seiberg duality is an involution

Mass perturbations and mass scales

6 Seiberg duality, improved

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