Introduction to cluster algebras and varieties

Vautouraude - Oct. 2019

Valdo Tatitscheff

valdo.tatitscheff@math.unistra.fr

IRMA, 7 rue René Descartes, 67084 Strasbourg Cedex

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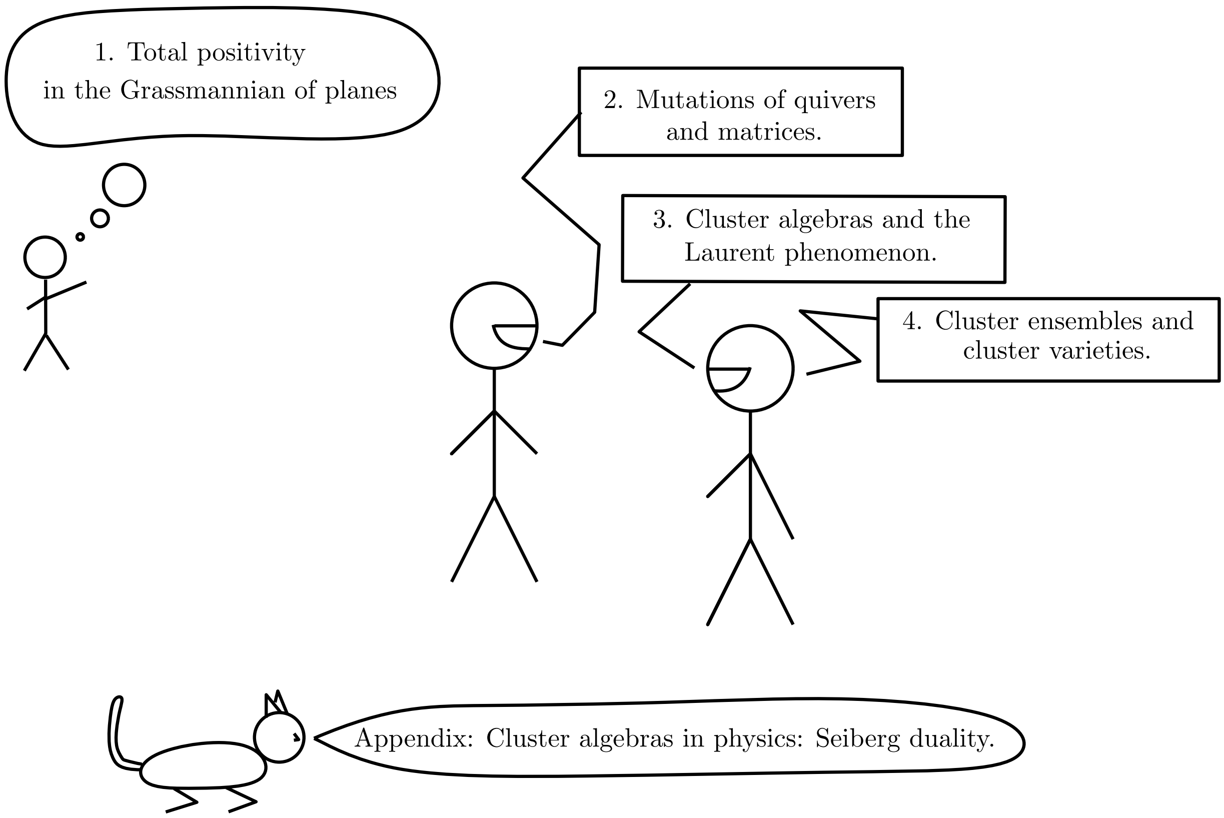
Introduction and narrative table of contents

Cluster algebras and varieties form one of these topics in maths that did not exist before the beginning of this millennium - still it has grown very fast and it is now a well-established field, just as other older subjects. The number and diversity of occurrences of cluster structures in maths and in physics is quite astonishing, given how ad-hoc the definitions seem, as one pass over them for the first time. The goal of this short-ish presentation is to motivate cluster algebras, Y-patterns, and cluster varieties, and then, to show some applications of the theory, and some examples in which such structures show their teeth. We mainly follow the lecture notes [FWZ] and [Mar14] in the first three sections. The last section - which introduces cluster ensembles - mainly follow their inventors' articles [FG09] and [FG07] (amongst others). The appendix is a condensate of many books, lecture notes (especially [Ber19]), and some personal thoughts.



1 Total positivity in the Grassmannian of planes

1.1 Cast of the characters

The *Grassmannian manifold* over $\mathbb{C}$ (denoted $\mathrm{Gr}_{k,m}$) is the complex manifold of k-dimensional subspaces of an m-dimensional complex vector space V. Let us fix a basis $(e_1, ..., e_m)$ of V (or equivalently, an isomorphism $V \to \mathbb{C}^m$). Thanks to it, any $k \times m$ matrix $[z]$ of rank k defines a point $[z] \in \mathrm{Gr}_{k,m}$.

Given a subset $J \subset \{1, ..., m\}$ with k-elements, the *Plücker coordinate* $P_J(z)$ is the $k \times k$ minor of z, determined by the column set J - note that the collection $(P_J(z))_{|J|=k}$ only depends on the row span of $[z]$, up to a common rescaling. This yields an embedding in the projective space $\mathbb{P}^{\binom{m}{k}-1}$ known as the *Plücker embedding*.

Definition 1. *The totally positive Grassmannian* $\mathrm{Gr}^+_{k,m}$ *is the subset of* $\mathrm{Gr}_{k,m}$ *consisting of points whose Plücker coordinates can be chosen so that all of them are in* $\mathbb{R}_+$ *(can be, since the coordinates are only defined up to a common rescaling).*

1.2 A recipe for efficient total positivity tests in $\mathrm{Gr}_2(\mathbb{C}^m)$

The problem. Let's now restrict to the case of $\mathrm{Gr}^+_{2,m} \subset \mathrm{Gr}_{2,m}$. The matrices $[z]$ under consideration are now $2 \times m$, and yield $\binom{m}{2}$ Plücker coordinates $P_{ij} := P_{\{i,j\}}$ labelled by pairs of integers $1 \leq i < j \leq m$.

How can one check if all the minors of a matrix $[z]$ *are positive?*

A priori (brute force) one needs to check $\binom{m}{2}$ minors. However, there are relations.

Example 1. *For* $1 \leq i < j < k < l \leq m$*, one has:*

$$P_{ik}P_{jl} = P_{ij}P_{kl} + P_{il}P_{jk} \ . \tag{1}$$

Eq. 1 implies for instance that there is no need to check the positivity of P_{ik} if one knows that $P_{jl}, P_{ij}, P_{kl}, P_{il}$ and P_{jk} are positive.

Are there efficient total positivity tests for the elements of $\mathrm{Gr}^+_{2,m}$*?*

A solution. Consider a convex m-gon P_m with vertices labelled clockwise. Let us assign the Plücker coordinate P_{ij} to the chord whose endpoints are i and j.

Let T be a triangulation of P_m by pairwise non-crossing diagonals but possibly common endpoints. The triangulation T can be seen as a maximal collection of non-crossing chords: as such it consists of m sides and $m-3$ diagonals, hence there is a distinguished collection $\tilde{x}(T)$ of $2m-3$ Plücker coordinates corresponding to the chords of T. This collection is called the *extended cluster* associated with T.

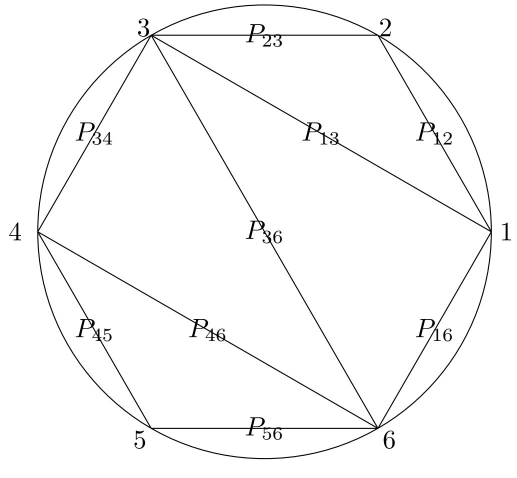


Figure 1: A triangulation T of a regular hexagon.

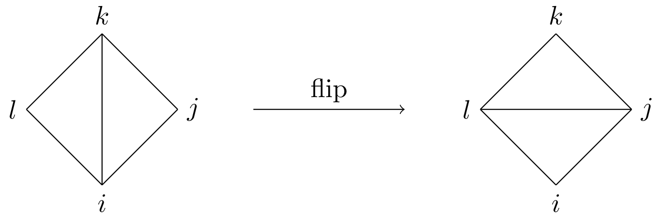
The Plücker coordinates corresponding to the sides of T are called *frozen variables*, or *coefficient variables*, while the remaining $m-3$ corresponding to diagonals of P_m are called *cluster variables*, and form a *cluster*. As can be checked, the $2m-3$ Plücker coordinates associated with the chords of any triangulation T are algebraically independent.

Theorem 1. *Any Plücker coordinate* P_{ij} *can be written as a subtraction-free rational expression in the elements of any given extended cluset* $\tilde{x}(T)$*. Thus, if the* $2m-3$ *Plücker coordinates* $P_{ij} \in \tilde{x}(T)$ *are positive at a given matrix* z*, then all the* 2×2 *minors of* z *are positive.*

Proof. First note that the relation of Eq. 1 strongly reminds the Ptolemy theorem, which states that in an inscribed quadrilateral, the products of the lengths of opposite sides add up to the product of the lengths of the two diagonals.

The proof of the theorem is based on the following three facts:

1. Any Plücker coordinate appears as an element of an extended cluster $\tilde{x}(T)$ for some triangulation of P_m.
2. Any two triangulations of P_m can be transformed into each other by a sequence of *flips*.



3. The flip involving the vertices i, j, k and l acts on the extended clusters by merely exchanging the Plücker coordinates P_{ik} and P_{jl}.

□

Remark 1. *Let T be a triangulation of the m-gon. It is fact that any Plücker coordinate can be written as a Laurent polynomial with positive coefficients in the Plücker coordinates appearing in $\tilde{x}(T)$. This is a special occurrence of the general Laurent phenomenon and positivity in cluster algebras (see Sec. 3 below).*

2 Mutations of quivers and matrices

Quivers (and more generally skew-symmetrizable matrices) encode, in a cluster algebra, the piece of information which generalises the combinatorics of triangulations and flips.

2.1 Quivers: definition and examples

Definition 2. *A* quiver *is a finite oriented graph, possibly with multiple arrows, but no loop nor oriented 2-cycle. It does not necessarily have to be connected. In the quivers we consider some vertices will be dubbed* frozen *(since they are mere spectators) and the other will be called* mutable*. Let us assume that there are no edge between two frozen vertices.*

Figure 2: These quiver are called the Markov quiver (left) and the Somos-4 quiver (right).

Let T be a triangulation of an m-gon P_m viewed it as an oriented surface (for instance, with its counter-clockwise orientation).One way to construct a quiver $\mathcal{Q}_T$ out of T in the following.

Assign a vertex of $\mathcal{Q}_T$ to each chord of T. Chords on the boundary of T define frozen vertices, while chords in the interior of T define mutable vertices. Inside each triangle of T, let us draw three arrows connecting consecutive (with respect to the counter-clockwise orientation) chords of the triangle under consideration - except if they are one the boundary. The triangulation of Fig. 1 yields the quiver drawn in Fig. 3, where numbers label mutable vertices and letters, the frozen ones.

2.2 Mutation of a quiver

There is an elementary transformation on quivers which generalizes the flips we saw earlier, called *mutation*. As the name suggests, mutation can only occur at a mutable vertex.

Definition 3. *Let k be a mutable vertex of a quiver $\mathcal{Q}$. The* mutation at k *yields a new quiver $\mu_k(\mathcal{Q})$, obtained after the following steps:*

1. *For every two-arrows path $i \to k \to j$, add an arrow $i \to j$ (unless i and j are frozen).*

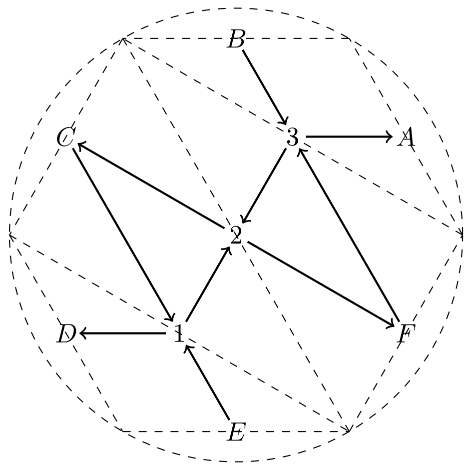


Figure 3: A quiver corresponding to the triangulation of 1

2. *Inverse the directions of all arrows incident to k.*

3. *Repeatedly remove oriented 2-cycles.*

Example 2. *1. The Markov quiver is invariant under the mutation at any of its vertices.*

2. *The mutation at the vertex* 1 *of the Somos*-4 *quiver at* 1 *yields a copy of the same quiver, rotated by* $90°$.

3. *One can check that this notion of flip indeed generalizes the* flips, *in the way that mutation at a mutable vertex of the quiver corresponding to a triangulation (constructed as described above) does correspond to a flip in the triangulation.*

Mutation class of a quiver. Let $\mathcal{Q}$ be a quiver, and $M_{\mathcal{Q}}$ the set of all quivers that one can get from $\mathcal{Q}$ by repeated mutations at mutable vertices. The relation $\mathcal{Q} \sim \mathcal{Q}'$ if $\mathcal{Q}$ and $\mathcal{Q}'$ are related through a sequence of mutations, is an equivalence relation. The set $M_{\mathcal{Q}}$ is the equivalence class of $\mathcal{Q}$ for $\sim$. In general, the set $M_{\mathcal{Q}}$ is infinite; but it may be finite. In this latter case one says that the quiver is of *finite mutation type*. The Markov quiver of Fig. 2.1 is an obvious example of such a finite mutation type quiver.

Frozen vertices do not interfere with being of finite mutation type or not. Quivers of finite mutation type without frozen vertices type have been completely classified in [FST12], and fall into three classes:

1. quivers with two vertices,

2. quivers associated with triangulations of punctured surfaces, as in Fig. 3,

3. 11 "exceptional" cases named $E_6, E_7, E_8, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}, X_6$ and X_7 (see Fig. 6.1 in [FST12]), and their mutations.

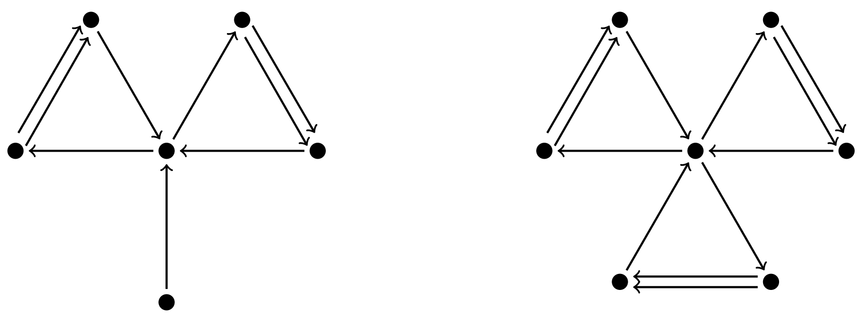


Figure 4: Exceptional quivers of finite mutation type X_6 (left) and X_7 (right).

2.3 Mutation of a skew-symmetrizable matrix

Quivers can (completely equivalently) be encoded in skew-symmetric matrices. Namely, let $\mathcal{Q}$ be a quiver with mutable vertices labelled $1, ..., n$ and frozen one, $n+1, ..., k_m$. The skew-symmetric matrix $\tilde{B}_{\mathcal{Q}}$ called the *extended* *exchange matrix* is the $m \times n$ matrix defined as $(\tilde{B}_{\mathcal{Q}})_{ij} = n$ if there are n arrows going from i to j in $\mathcal{Q}$, or $-n$ if there are n arrows going from j to i in $\mathcal{Q}$, and 0 otherwise.

The *exchange matrix* $B_{\mathcal{Q}}$ is the upper $n \times n$ sub-matrix of $\tilde{B}_{\mathcal{Q}}$.

Example 3. *The following three matrices are the extended exchange matrices of the Markov quiver and the Somos-*4 *quiver (given in Fig.* 2.1*), and the quiver associated with the triangulation of the* 6*-gon consider in* 3.

$$\tilde{B}_{\text{Markov}} = \begin{bmatrix} 0 & 2 & -2 \\ -2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix} \quad \tilde{B}_{\text{Somos}-4} = \begin{bmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & 1 & -2 \\ -2 & -1 & 0 & 3 \\ 1 & 2 & 0 & 4 \end{bmatrix} \quad \tilde{B}_{\text{Fig.3}} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{2}$$

Proposition 1. *Let k be a mutable vertex of a quiver $\mathcal{Q}$ with extended exchange matrix $\tilde{B}_{\mathcal{Q}} = (b_{ij})$. The extended exchange matrix $\tilde{B}' = \tilde{B}_{\mu_k(\mathcal{Q})} = (b'_{ij})$ of the quiver $\mu_k(\mathcal{Q})$ is given by*

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k \ , \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0 \ , \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0 \ , \\ b_{ij} & \text{otherwise} \ . \end{cases} \tag{3}$$

Mutations of skew-symmetric matrices can be generalised to the case of skew-symmetrizable matrices.

Definition 4. *An $n \times n$ matrix B with integer entries is said to be skew-symmetrizable if there exists an n-uplet of integers $(d_1, ..., d_n)$ such that $d_i b_{ij} = -d_j b_{ji}$. An extended skew-symmetrizable matrix is an $m \times n$ matrix ($m \geq n$) such that its top $n \times n$ sub-matrix is skew-symmetrizable.*

Definition 5. *Let $\tilde{B} = (b_{ij})$ be an extended skew-symmetrizable $m \times n$ matrix (with $m \geq n$), and let $k \in \{1, ..., n\}$. The mutation of $\tilde{B}$ at k is the extended skew-symmetrizable matrix $\mu_k(\tilde{B})$ with entries satisfying Eq.* 3.

Example 4. *Mutations of skew-symmetrizable matrices have the following properties:*

1. *The mutated matrix is indeed skew-symmetrizable (with the same choice of $(d_1, ..., d_n)$).*
2. $\mu_k(\mu_k(\tilde{B})) = \tilde{B}$.
3. *If $b_{ij} = b_{ji} = 0$ then $\mu_i(\mu_j(\tilde{B})) = \mu_j(\mu_i(\tilde{B}))$.*

Skew-symmetrizable matrices can be encoded in a so-called *generalized quiver*, where vertices carry the *multipliers* d_i, and with $d_i b_{ij}$ arrows from i to j if this number is positive, or $-d_i b_{ij} = d_j b_{ji}$ arrows from j to i otherwise.

3 Cluster algebras and the Laurent phenomenon

Cluster algebras (of geometric type) form a class of commutative rings introduced by Fomin and Zelevinski in a series of papers ([FZ02], [FZ03], [FZ05], [FZ07]). A cluster algebra A of rank n is an integral domain with distinguished subsets of size m called *clusters*, the union of which generates A.

3.1 Definition of cluster algebras of geometric type

Let $m \geq n$ be two integers. The *ambient field* of a (complex) cluster algebra is the field $\mathcal{F} = \mathbb{C}(x_1, ..., x_m)$ of complex rational fractions in m variables.

Definition 6. *A* (labelled) seed *of geometric type in $\mathcal{F}$ is a pair $(\tilde{x}, \tilde{B})$, where $\tilde{x} = (x_1, ..., x_m)$ is an m-tuple of elements of $\mathcal{F}$ forming a free generating set, and $\tilde{B}$ is an extended skew-symmetrizable matrix of size $m \times n$. The* cluster *of the seed is the m-tuple $\tilde{x}$. The x_i for $i = 1, ..., n$ are called* cluster variables*, while the x_i for $i = n+1, ..., m$ are the* frozen variables.

Definition 7. *Let $(\tilde{x}, \tilde{B})$ be a seed, and let $k \in \{1, ..., n\}$. The mutation of $(\tilde{x}, \tilde{B})$ in the direction k is another seed $\mu_k(\tilde{x}, \tilde{B}) = (\tilde{x}', \tilde{B}')$, where $\tilde{B}' = \mu_k(\tilde{B})$ and where $\tilde{x}' = (x'_1, ..., x'_m)$ is given by:*

$$\begin{cases} x'_i & = x_i \text{ for } i \neq k \\ x_k x'_k & = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}} \end{cases} \ . \tag{4}$$

Let T_n be the n-valent regular tree, and label its edges by $1, ..., n$ in such a way that the n edges incident to a vertex all have different labels.

Definition 8. *A seed pattern on T_n is an assignment $(\tilde{x}(t), \tilde{B}(t))_{t\in T_n}$ of a labelled seed to each vertex of T_n, in such a way that the seeds corresponding to two neighbouring vertices linked by an edge carrying the label k are obtained from each other by mutation in the direction k.*

Let χ be the set of all cluster variables appearing in a seed pattern $(\tilde{x}(t), \tilde{B}(t))_{t\in T_m}$, and let $R = \mathbb{C}[x_{n+1}, ..., x_m]$ be the ring generated over $\mathbb{C}$ by the frozen variables $x_{n+1}, ..., x_m$.

Definition 9. *The cluster algebra $\mathcal{A}$ of geometric type associated with the seed pattern $(\tilde{x}(t), \tilde{B}(t))_{t\in T_m}$ is the R-subalgebra of $\mathcal{F}$ generated by all cluster variables: $A = R[\chi]$.*

Example 5. *1. The tree T_1 has exactly two vertices. There are only two seeds and two* clusters*: (x_1) and (x_1'). The extended exchange matrix $\tilde{B}$ is an column with m entries (the first one is necessarily 0 by antisymmetry). The relation in Eq. 4 becomes*

$$x_1 x_1' = M_1 + M_2 \ , \tag{5}$$

where M_1 and M_2 are monomials in the frozen variables. The cluster algebra is the $\mathbb{C}[x_2, ..., x_m]$-subalgebra of $\mathcal{F} = \mathbb{C}(x_1, ..., x_m)$ generated by x_1 and x_1' subject to Eq. 5.

2. Any 2×2 skew-symmetrizable matrix has the form

$$\pm B(b,c) = \pm \begin{bmatrix} 0 & b \\ -c & 0 \end{bmatrix} ,$$

where b and c are two integers that are either strictly positive or equal to zero. A mutation applied to a matrix of this form merely changes its sign. Let $\mathcal{A}(b,c)$ be the cluster algebra corresponding to the seed pattern obtained form the matrix above. The case $b = c = 0$ is not very interesting, hence we assume $b, c > 0$.

Let us start with a seed $((x_1, x_2), B(b,c))$. A mutation in the direction 2 yields $((x_1, x_2' =: x_3), -B(b,c))$. Mutating again (this time in the direction 1), one gets $((x_4 := x_1', x_3), B(b,c))$, and so on, and so forth. The cluster variables hence fit in a sequence $(..., x_1, x_2, x_3, ...)$. The exchange relation is:

$$x_{k-1}x_{k+1} = \begin{cases} x_k^c + 1 \ , & k \text{ even} \\ x_k^b + 1 \ , & k \text{ odd} \end{cases} \tag{6}$$

(a) In the case of $\mathcal{A}(1,1)$ (called the cluster algebra of type A_2), Eq. 6 gives the following 5-periodic sequence of cluster variables:

$$..., x_1, x_2, \frac{x_2+1}{x_1}, \frac{x_1+x_2+1}{x_1 x_2}, \frac{x_1+1}{x_2}, x_1, x_2, ... \ .$$

(b) In the cases of $\mathcal{A}(1,2)$ (type B_2) and of $\mathcal{A}(1,3)$ (type G_2), one can easily see that the sequence is periodic as well.

(c) The calculation of the first few terms of the sequence in the case of $\mathcal{A}(1,3)$, when restricting to $x_1 = x_2 = 1$, yields:

$$1, 1, 2, 3, 41, 937, 67, 21506, 321, 493697, ... \ , \tag{7}$$

and it can be proven that the sequence is not periodic.

Remark 2. *If the sequence of cluster variables is periodic one says that the corresponding cluster algebra is of finite type. This very specific type of cluster algebras has been completely classified in [FZ03]. Cluster algebras of finite type are in $1:1$ correspondence with the Dynkin diagram of finite dimensional simple Lie algebras.*

Note that the terms appearing in 7 are all integers... while looking back at the recurrence equations 6, there were a priori no reason for it to be the case! This is called the Laurent phenomenon, and is a central and very interesting feature of cluster algebras.

3.2 The Laurent phenomenon

Theorem 2. *In a cluster algebra of geometric type, each cluster variable can be expressed as a Laurent polynomial with integer coefficients in the elements of any extended cluster. Moreover, frozen variables do not appear in the denominators of these Laurent polynomials.*

Idea of the proof. Let $t_0 \in T_n$ an (arbitrary) initial vertex, corresponding to the extended cluster $(\tilde{x}_0, \tilde{B}_0)$, with $\tilde{x}_0 = (x_1, ..., x_m)$ and $\tilde{B}_0 = (b_{ij})_{m\times n}$. Let $t \in T_n$ be an (arbitrary) vertex, and $x \in x(t)$ a cluster variable at t. Our goal is to prove that t can be expressed as a Laurent polynomial in the cluster variables in $\tilde{x}_0$.

To that purpose, let

$$t_0 \overset{j}{\text{———}} t_1 \overset{k}{\text{———}} t_2 \text{———} \cdots \text{———} t \tag{8}$$

be the unique path in T_n connecting t_0 to t, and let d be its length. The proof is an induction on d.

Initialisation: $(d = 1, 2)$ trivial, from the mutation formulae.

Heredity: one can distinguished two cases, whether $b_{jk} = b_{kj} = 0$ or $b_{jk}b_{kj} < 0$. The second case is harder and longer, even if the ideas are more or less the same in both cases. Subsequently we are only going to prove the first one here in order to give a taste of the proof without reaching a pain threshold, and refer to [FWZ] for the rest of the proof.

Hence let us assume now, that we are in the following situation:

$$t_3 \overset{k}{\text{———}} t_0 \overset{j}{\text{———}} t_1 \overset{k}{\text{———}} t_2 \quad . \tag{9}$$

By assumption, at t_0 one has $\mu_j \circ \mu_k = \mu_k \circ \mu_j$, hence each of the seeds at t_1 and t_3 is at a distance $d-1$ away from a seed containing x. By induction, x is Laurent polynomial in $\tilde{x}(t_1) = (x_1, ..., x'_j, ..., x_m)$ - where $x'_j = \frac{M_1+M_2}{x_j}$ with M_1 and M_2 two polynomials in $(x_1, ..., x_m)$, and also in $\tilde{x}(t_3) = (x_1, ..., x'_k, ..., x_m)$ - where $x'_k = \frac{M_3+M_4}{x_k}$, with M_3 and M_4 two polynomials in $(x_1, ..., x_m)$.

All the possible non-monomial factors in the expression of x in terms of the variables in $\tilde{x}(t_0)$ have to come from $M_1 + M_2$ (resp. $M_3 + M_4$) in the first (resp. second) computation. Were $M_1 + M_2$ and $M_3 + M_4$ coprime, these non-monomial factors would necessarily be trivial, however this co-primality condition does not hold in general.

Hence let us ass a frozen variable x_{m+1} in such a way that $b_{j,m+1} = 1$ and all other $b_{\cdot,m+1} = 0$. Then $M_1 + M_2$ is a binomial which as degree 1 in x_{m+1}, hence it is irreducible. It can moreover not divide $M_3 + M_4$ since the latter does not depend on x_{m+1}. Hence the argument go through, and we retrieve the case we are interested in by specializing x_{m+1} to 1.

□

3.3 A few applications in Number Theory

3.3.1 The Somos-4 sequence

Consider the Somos-4 quiver given in Fig. 2.1, and recall the result 2. in Ex. 2. Let us start with a seed $(\tilde{x}, \tilde{B})$ where each cluster variable x_1, x_2, x_3, x_4 corresponds to a vertex of the Somos-4 quiver, and with exchange matrix corresponding to the latter. Mutation in the direction 1 yields the seed $(\tilde{x}', \tilde{B}')$, where $\tilde{x}' = (x_5 := x'_1, x_2, x_3, x_4)$ satisfies:

$$x_5 x_1 = x_2 x_4 + x_3^2 \ , \tag{10}$$

and where $\tilde{B}'$ is exactly $\tilde{B}$, up to permutation of the indices of its rows and columns. Namely, if one permutes according to $i' \leftarrow i + 1$, then the seeds $((x_2, x_3, x_4, x_5), \tilde{B}')$ and $(\tilde{x}, \tilde{B})$ are exactly the same. This procedure can be repeated again and again, yielding $x_6, x_7, ...$ and eventualky a sequence whose recurrence relation generalizes Eq. 10:

$$x_{n+4} x_n = x_{n+1} x_{n+3} + x_{n+2}^2 \ , \tag{11}$$

The Somos-4 sequence is defined as the sequence satisfying Eq. 11, with initial condition $x_1 = x_2 = x_3 = x_4 = 1$.

Remark 3. *Theorem* 2 *implies that all the terms in this sequence are in fact integers since there are Laurent polynomials in* x_1, x_2, x_3, x_4.

3.3.2 Euler's counterexample of the primality of Fermat numbers

A long time ago Fermat conjectured that all the numbers of the form $F_n = 2^{2^n} + 1$ are primes. This was shown to be false by Euler in 1732, since F_5 decomposes as:

$$F_5 = 2^{32} + 1 = 641 \cdot 6700417 \ .$$

This result can be presented as a corollary of Theorem 2, as follow. Consider the initial seed:

$$\tilde{\sigma} = \left(\tilde{x} = (x_1, x_2, x_3), \tilde{B} = \begin{bmatrix} 0 & 4 \\ -1 & 0 \\ 1 & -3 \end{bmatrix} \right) \ . \tag{12}$$

We are going to specialize to $(x_1, x_2, x_3) = (3, -1, 16)$. Mutation in the direction 1 yields a new cluster variable x'_1 which satisfies $x'_1 x_1 = x_2 + x_3$, hence (x'_1, x_2, x_3) specializes to $(5, -1, 16)$. Theorem 2 then states that any cluster variable that can be obtained by repeated mutations on one of these two seeds is an integer, possibly divided by a power of 5, AND an integer, possibly divided by a power of 3, hence it has to be an integer. Now, one can check that the seed $\mu_1\mu_2\mu_1\tilde{\sigma}$ is:

$$\left(\tilde{x} = (-128, -641, 16), \tilde{B} = \begin{bmatrix} 0 & -4 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}\right) , \tag{13}$$

hence applying μ_2 once again, we get a new cluster variable x such that $-641x = 2^{7\cdot 4} \cdot 16 + 1 = F_5$, proving Euler's result.

3.3.3 Markov's numbers and unicity conjecture

Consider again Markov's quiver on the left of Fig. 2.1. Since it is invariant under mutations, exchange relations will always be of the form $x'_i x_i = x_j^2 + x_k^2$, for all possible choices of i, j and k such that $\{i, j, k\} = \{1, 2, 3\}$. Observe that if a cluster (x_1, x_2, x_3) satisfies the (Diophantine) equation

$$3x_1x_2x_3 = x_1^2 + x_2^2 + x_3^2 , \tag{14}$$

then the image of this cluster under any mutation is also a solution. If one starts with the specialization $(1, 1, 1)$, one gets an (a priori) infinite sequence of triples of integers that all satisfy this equation.

Eq. 14 is called the *Markov equation*, and a triple which is solution is called a Markov triple. It can be shown that any Markov triple appears as the image, under multiple mutations, of the original cluster $(1, 1, 1)$.

Markov's uniqueness conjecture is that, given any Markov number c (an integer appearing in a triple solution to Eq. 14), there is a unique triple, solution to Eq. 14, whose maximal element is c (up to permutations).

3.4 Y-patterns

The following result is about Laurent polynomials that appear as the ratio of the two terms on the right-hand side of exchange relations (Eq. 4). An interesting fact is that the way these polynomials change under mutation only depends on the exchange matrix and not on the extended one.

Let $(\tilde{x} = (x_1, ..., x_n, x_{n+1}, ..., x_m), \tilde{B} = (b_{ij}))$ and $(\tilde{x}' = (x'_1, ..., x'_n, x'_{n+1}, ..., x'_m), \tilde{B}' = b'_{ij})$ be two extended seeds related by mutation at k. We define the tuples $\tilde{y} = (y_1, ..., y_n)$ and $\tilde{y} = (y'_1, ..., y'_n)$ by:

$$y_j = \prod_{i=1}^{m} x_i^{b_{ij}}, \;\; y'_j = \prod_{i=1}^{m} (x'_i)^{b_{ij}} \tag{15}$$

Proposition 2. *One has the following transformations properties for the y's:*

$$y'_j = \begin{cases} y_k^{-1} & \text{if } k = j, \\ y_j(y_k + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leq 0 , \\ y_j(y_k^{-1} + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geq 0 . \end{cases} \tag{16}$$

Proof. Consider the case $j = k$. Then

$$y'_k = \prod_i (x'_i)^{b'_{ik}} = \prod_{i \neq k} x_i^{b'_{ik}} = \prod_{i \neq k} x_i^{-b_{ik}} = y_k^{-1} \tag{17}$$

What remains of the checks are likewise easy calculations. □

Definition 10. *A Y-seed of rank n in a field $\mathcal{F}$ is a pair (Y, B), with Y an n-tuple of elements of $\mathcal{F}$, and B is a skew-symmetrizable $n \times n$ matrix (with integer entries). Two Y-seeds (Y, B) and (Y', B') are said to be related through a mutation in the direction k if $B' = \mu_k(B)$ and Y' and Y are related via Eq.* 16.

Definition 11. *A Y-pattern of rank n is a collection of Y-seeds $(Y(t), B(t))_{t\in T_n}$ labelled by the vertices of an n-regular tree, such that the Y-seeds on two neighbouring vertices linked by an edge carrying the label k are related through a mutation in the direction k.*

There are some subtleties concerning Y-patterns but we cowardly refer instead to [FWZ] instead of spending time on that matter ourselves. The definition of cluster ensembles (and varieties) by V. Fock and G. Goncharov, to which we now turn, includes an object (the $\mathcal{X}$-space) which resembles Y-patterns very much.

4 Cluster ensembles and cluster varieties

4.1 Motivations

Cluster algebras surely have a lot to do with total positivity. In [FG09], V. Fock and A. Goncharov defined and studied the properties of *cluster ensembles*, which form some kind of algebro-geometric counterpart to the algebraic structures defined in the previous section.

Instead of presenting the definitions in all generalities as it is the case in [FG09], we will restrict to the complex case.

The multiplicative algebraic group over $\mathbb{C}$ (denoted $\mathbb{G}_m(\mathbb{C})$) is the affine algebraic group $\mathbb{C}^\times$ (with the Zariski topology). The ring of regular functions on $\mathbb{G}_m(\mathbb{C})$ is $\mathbb{C}[x, x^{-1}]$. A product of multiplicative groups (over $\mathbb{C}$) $\mathbb{G}_m(\mathbb{C})^n = \mathbb{C}^n$ is called a *split algebraic torus* (over $\mathbb{C}$). It is an abelian group and an algebraic variety, and the two structure are compatible (in the sense that the group operations are morphisms of algebraic varieties).

A split algebraic torus H has special regular functions called *characters*, which are the group morphisms from H to $\mathbb{C}^\times$. Elements of $\mathrm{Hom}(\mathbb{C}^\times, H)$ are referred to as *cocharacters*.

Definition 12. *A rational function $f \in \mathbb{C}(x, x^{-1})$ is* positive *if it belongs to the semi-field generated by the characters of H. Equivalently, it is positive if it can be written $f = f_1/f_2$ for f_1 and f_2 two linear combinations of characters with positive integral coefficients. A positive rational map between two split tori H_1 and H_2 is a rational map $f : H_1 \to H_2$ such that f^* induces a homomorphism of semifields of positive rational functions.*

Let Pos be the category where objects are split algebraic tori and maps are positive rational maps.

Definition 13. *A groupoid is a category where all morphisms are isomorphisms (assuming that the set of morphisms between any two objects is not empty). The fundamental group of a groupoid is the automorphism group of any object in the category (it is well-defined only up to conjugation).*

Definition 14. *Let $\mathcal{G}$ be a groupoid. A positive space is a functor*

$$\psi : \mathcal{G} \to \mathrm{Pos}\ . \tag{18}$$

Remark 4. *This definition of positive spaces makes it possible to speak of points of a positive space with values in a semi-field. Indeed everything we have just done can be defined not only over $\mathbb{C}$ but over any ring, and also over semi-fields such as the tropical ones.*

In pedestrians terms, we defined geometric objects using algebraic operations that only involve adding, multiplying and taking quotients, but never subtracting. These objects can therefore be defined *over any algebraic structure where these operations exist*. Any ring (even $\mathbb{Z}$) works, however the rational functions on the corresponding object will be naturally defined on a field of fractions over $\mathbb{Q}$.

A semi-field is a set P equipped with operations of addition and multiplication, such that addition is commutative and associative, multiplication makes P an abelian group, and multiplication is distributive with respect to addition. For instance:

- the set $\mathbb{R}_+$ with usual addition and multiplication,
- the tropical semi-field associated with $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$, respectively denoted $\mathbb{Z}^t$, $\mathbb{Q}^t$ and $\mathbb{R}^t$, with multiplication $a \otimes b = a + b$ the usual addition and addition the max, that is, $a \oplus b = \max(a, b)$.

Given a split torus H (here consider over $\mathbb{Z}$) and a semi-field P, the set of P-valued points of H is defined as

$$H(P) := X_*(H) \otimes_{\mathbb{Z}} P\ , \tag{19}$$

with $X_*(H)$ the group of cocharacters of H and the tensor product is with the abelian group defined by the semi-field P.

Definition 15. *Given a positive space $\mathcal{X}$, there is a unique set $\mathcal{X}(P)$ of P-points of H, defined as the disjoint union of the P-points of the split tori in the image of the functor $\mathcal{X}$, modulo identifications given by the morphisms of the groupoid.*

Definition 16. *An important point of Eq.* 18 *is that through the functoriality of a positive space, every isomorphism $\alpha \to \beta$ in the groupoid induces a* birational isomorphism $H_\alpha \dashrightarrow H_\beta$*, i.e. an actual isomorphism between an open dense subset of $H\alpha$ and an open dense subset of H_β.*

Subsequently one can glue these algebraic split tori (the H_α) of the same rank (because of the isomorphisms) together, along open dense subsets. Starting with a positive space $\mathcal{X}$, the result of this procedure gives a prescheme $\mathcal{X}^*$ (not a scheme because it is not necessarily separable).

4.2 Definition of the cluster ensembles

4.2.1 Seeds and cluster tori

Definition 17. *A seed is a quadruple* $(\Lambda, (*,*), \{e_i\}, \{d_i\})$*, where:*

i) Λ *is a lattice (a free abelian group),*

ii) $(*,*)$ *is a skew-symmetric* $\mathbb{Q}$*-valued bilinear form on* Λ*,*

iii) $I = \{e_i\}$ *is a basis of the lattice* Λ*, and* I_0 *is a subset of* I *containing the frozen basis vectors,*

iv) $\{d_i\}$ *are positive integers assigned to the basis vectors, such that* $\epsilon_{ij} := (e_i, e_j)d_j \in \mathbb{Z}$ *unless* $i, j \in I_0 \times I_0$*.*

A lattice Λ gives rise to a split algebraic torus $\mathcal{X}_\Lambda = \mathrm{Hom}(\Lambda, \mathbb{G}_m)$. An element $v \in \Lambda$ provides a character X_v whose value on a $x \in \mathcal{X}_\Lambda$ is $x(v)$. The category of finite rank lattices and the category of split algebraic tori are dual to each other.

An element a of the dual lattice Λ gives rise to a cocharacter $\phi_a : \mathbb{G}_m \to \mathrm{Hom}(\Lambda, \mathbb{G}_m)$. If F is a field, on the level of F points $\phi_a(f)$ is the homomorphism $v \to f^{a(v)}$.

Definition 18. *The seed* $\mathcal{X}$*-torus is the split algebraic torus* $\mathrm{Hom}(\Lambda, \mathbb{G}_m)$*. It carries a natural Poisson structure provided by the form* $(*,*)$*:*

$$\{X_v, X_w\} = (v, w) X_v X_w \tag{20}$$

The basis $\{e_i\}$ *provides cluster* $\mathcal{X}$*-coordinates* $\{X_i\}$*. The latter form a basis in the group of characters of* $\mathcal{X}_\Lambda$*.*

Let $\{f_i\}$ be the *quasi-dual basis* defined by $f_i = d_i^{-1} e_i$ for all i, and let $\Lambda^\circ \subset \Lambda^* \otimes_{\mathbb{Z}} \mathbb{Q}$ be the lattice generated by the f_i.

Definition 19. *The seed* $\mathcal{A}$*-torus is the split algebraic torus* $\mathcal{A}_\Lambda = \mathrm{Hom}(\Lambda^\circ, \mathbb{G}_m)$*. The basis* $\{f_i\}$ *provides cluster* $\mathcal{A}$*-coordinates* $\{A_i\}$*. There is a natural closed two-forms* Ω *on the torus* $\mathcal{A}_\Lambda$*.*

The two structure are of course not unrelated, since the following map of lattices:

$$\begin{array}{rcl} p^* : \Lambda & \to & \Lambda^\circ \\ v & \mapsto & \sum_j (v, e_j) e_j^* = \sum_j (d_j (v, e_j)) f_j \end{array} , \tag{21}$$

yields, by the isomorphism of categories mentioned above, a homomorphism of seed tori $p : A_\Lambda \to \mathcal{X}_\Lambda$.

Proposition 3. *The fibers of* p *are the leaves of the null-foliation of the closed* 2*-form* Ω*. The image* $p(\mathcal{A}_\Lambda)$ *is a symplectic leaf of the Poisson structure on* $\mathcal{X}_\Lambda$*, and the symplectic structure on* $p(\mathcal{A}_\Lambda)$ *induced by* Ω *coincides with the one induced by restriction of the Poisson structure.*

4.2.2 Cluster transformations

Let us denote $\epsilon_{ij} = d_j \cdot (e_i, e_j)$. It is a non-symmetric bilinear form on Λ called the *exchange function.* Lots of objects evoked above admit quite nice expressions with the help of this exchange function; it is the case of the Poisson structure on the $\mathcal{X}_\Lambda$ torus, for instance:

$$\{X_i, X_j\} = \frac{\epsilon_{ij}}{d_j} X_i X_j \tag{22}$$

and also of the 2-form Ω on $\mathcal{A}_\Lambda$:

$$\Omega = \sum_{i,j \in I} d_i \epsilon_{ij} d \log A_i \wedge d \log A_j \tag{23}$$

Eventually, the homomorphism of Eq. 21 satisfies:

$$p^* X_i = \prod_{j \in I} A_j^{\epsilon_{ij}} . \tag{24}$$

For it is our destiny, let's state that there is a notion of mutation that maps a seed to a seed, and acts on the exchange function as follows:

$$\epsilon'_{ij} = \begin{cases} -\epsilon_{ij} & \text{if } k \in \{i, j\} , \\ \epsilon_{ij} & \text{if } \epsilon_{ik}\epsilon_{kj} \le 0, \ k \ne \{i, j\} , \\ \epsilon_{ij} + |\epsilon_{ik}| \epsilon_{kj} & \text{if } \epsilon_{ik}\epsilon_{kj} > 0, \ k \ne \{i, j\}. \end{cases} \tag{25}$$

This transformation is involutive on the exchange function (but not necessarily on the basis, see [FG09]).

Any seed mutation in the direction k induces a positive rational map between the corresponding seed $\mathcal{X}$- and $\mathcal{A}$- tori. Let us denote X'_i (resp. A'_i) the cluster coordinates on the seed torus $\mu_k(\mathcal{X}_\Lambda)$ (resp. $\mu_k(\mathcal{A}_\Lambda)$), and define:

$$\mu_k^* X'_i := \begin{cases} X_k^{-1} & \text{if } i = k \ , \\ X_i(1 + X_k^{\mathrm{sgn}(\epsilon_{ik})})^{-\epsilon_{ik}} & \text{if } i \neq k \ . \end{cases} \tag{26}$$

$$A_k \cdot \mu_k^* A'_k = \prod_{j:\epsilon_{kj}>0} A_j^{\epsilon_{kj}} + \prod_{j:\epsilon_{kj}<0} A_j^{-\epsilon_{kj}} \ ; \ \mu_k^* A'_i = A_i \text{ for } i \neq k \ . \tag{27}$$

A *seed cluster transformation* is a composition of seed isomorphisms and mutations. It gives rise to a *cluster transformation* of the corresponding seed $\mathcal{X}$- and $\mathcal{A}$- tori. We define the cluster modular groupoid $\mathcal{G}_{|\sigma|}$ as the category whose objects are all the seeds equivalent to a given one σ, and where morphisms are the cluster transformations modulo the trivial ones. Eventually we have two positive spaces $\mathcal{G}_{|\sigma|} \to \mathcal{X}_{|\sigma|}$ and $\mathcal{G}_{|\sigma|} \to \mathcal{A}_{|\sigma|}$, sharing the same coordinate groupoid.

4.3 Properties of cluster ensembles

Cluster ensembles have been studied by V. Fock and A. Goncharov in a series of articles, starting with the seminal [FG06]. They enjoy very interesting properties.

- Any cluster $\mathcal{X}$-variety admit a canonical non-ambiguous deformation quantization:

  $$\{\cdot,\cdot\} \to [\cdot,\cdot] \text{ s.t. } [f,g] = f \star g - g \star f \tag{28}$$

  for some deformed product $\star$.
- The 2-form Ω on any cluster $\mathcal{A}$-variety admits a motivic avatar.
- There are duality conjectures which roughly goes as follows. The positive regular functions on a cluster $\mathcal{X}$- (resp. $\mathcal{A}$-) variety form a cone, whose generators are parametrised by the corresponding tropical $\mathcal{A}$- (resp $\mathcal{X}$-) variety.

4.4 The moduli spaces $\mathcal{M}_{0,n}$ and $\mathrm{Gr}_{2,m}$

Recall that the automorphisms of the Riemann sphere $\mathbb{P}^1(\mathbb{C})$ form the group $\mathrm{PSL}_2(\mathbb{C})$ called the Möbius group.

Definition 20. *The quotient of $\mathbb{P}^1(\mathbb{C})^n$ by the diagonal action of the Möbius group is called the configuration space of n points on $\mathbb{P}^1(\mathbb{C})$, or moduli space of n points on $\mathbb{P}^1(\mathbb{C})$, and is denoted $\mathcal{M}_{0,n}$.*

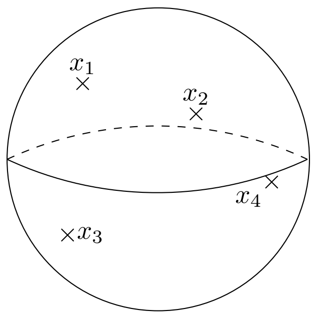


Figure 5: Four points on $\mathbb{P}^1(\mathbb{C})$.

The Riemann sphere is built as the compactification of $\mathbb{C}$ with a point at infinity, denoted ∞. This description yields a coordinate system on $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$, on which $\mathrm{PSL}_2(\mathbb{C})$ acts as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ z \ \mapsto \ \tfrac{az+b}{cz+d} \quad . \tag{29}$$

The group $\mathrm{PSL}_2(\mathbb{C})$ is 3-transitive: for any triple x_1, x_2 and x_3 of distinct points on $\mathbb{P}^1(\mathbb{C})$, there is a unique transformation in $\mathrm{PSL}_2(\mathbb{C})$ sending the triple (x_1, x_2, x_3) to $(0, 1, \infty)$ (or any other reference triple).

Starting with four points on the sphere, and assuming that three of them are distinct, there is a transformation in $\mathbb{P}^1(\mathbb{C})$ sending the latter to $0, 1$, and ∞ respectively. The position of the fourth point is an invariant of the configuration, the *cross-ratio* of four-points. It is the only invariant. If the four initial points have coordinate z_1, z_2, z_3 and z_4 in $\mathbb{C} \cup \infty$, the cross-ratio of the (ordered) quadruple can be expressed as

$$r(x_1, x_2, x_3, x_4) = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_2 - x_3)(x_4 - x_1)} \ . \tag{30}$$

The cross-ratio gives an isomorphism between the moduli space of four *ordered* points on $\mathbb{P}^1(\mathbb{C})$, and $\mathbb{P}^1(\mathbb{C})$.

In general, there is an action of the permutation group $\mathfrak{S}_n$ on the moduli space of n points $\mathcal{M}_{0,n}$. As soon as one chooses a cyclic ordering of the points, this permutation group breaks down to $\mathbb{Z}_n$. Let us put the cyclically ordered n points on a circle, in such a way that the cyclic orientation coincides with the counter-clockwise orientation of the circle. Let us choose a triangulation of the resulting n-gon, and assign to each edge the cross-ratio of the four vertices (read in a counter-clockwise order) of the quadrilateral of which it is a diagonal, starting from one of the ends of the diagonal. This gives a parametrization of the moduli space of n ordered points on $\mathbb{P}^1(\mathbb{C})$.

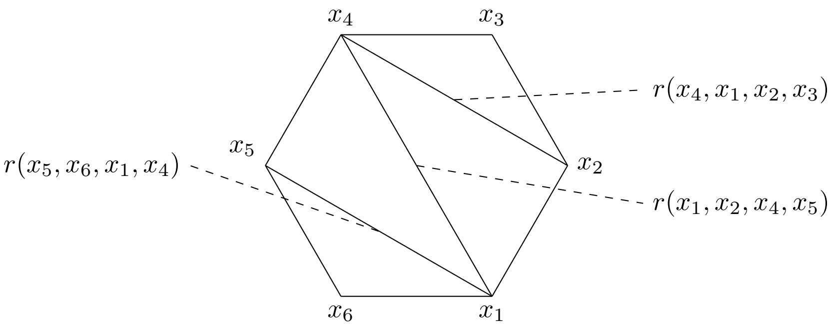


Figure 6: A parametrization of the moduli space of six ordered points

Under a flip, these coordinates change as shown in Fig. 7. These transformations are called $\mathcal{X}$-mutation formulae.

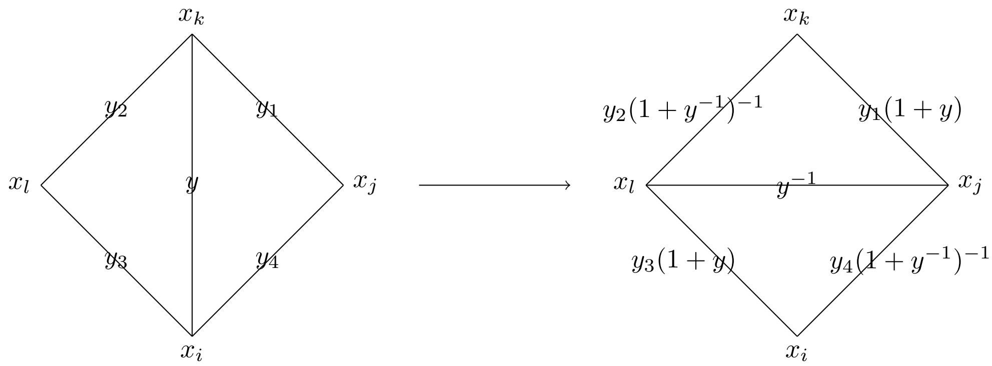


Figure 7: The $\mathcal{X}$ transformation rules.

4.5 (Classical) Teichmüller spaces of type $\mathcal{X}$ and $\mathcal{A}$

Let $\Sigma_{g,n}$ be a topological surface obtained from the compact one of genus g be removing n points. An *ideal triangulation* of $\Sigma_{g,n}$ is a maximal collection of isotopy classes of non-intersecting and non-self-intersecting arcs with endpoints the punctures.

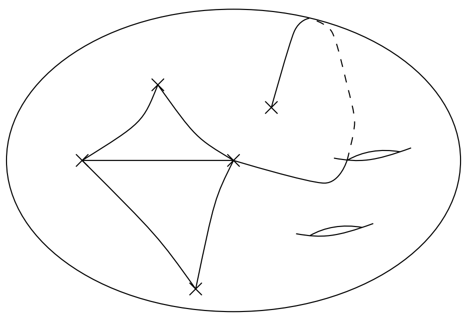


Figure 8: Part of an ideal triangulation of $\Sigma_{2,5}$.

The topology of $\Sigma_{g,n}$ constraints the possible triangulations. There are n punctures and any triangulation forms a cellular decomposition of $\Sigma_{g,n}$ in which every 2-cell is a triangle. The Euler characteristic of $\Sigma_{g,n}$ can be computed in terms of the triangulation data, and also in terms of the genus and n. That yields the result that every triangulation of $\Sigma_{g,n}$ has $6g-6+3n$ edges. Similarly, the number of faces is always $4g-4+2n$.

Definition 21. *The Teichmüller space of $\Sigma_{g,n}$ is the space of hyperbolic metrics (equivalently, complex structures - this is Riemann's uniformisation theorem) on $\Sigma_{g,n}$, modulo all the diffeomorphisms in the connected component of identity.*

Hyperbolic geometry enjoys some very nice properties such as the following.

Proposition 4. *In any compact hyperbolic manifold, any non-trivial free homotopy class admits a unique geodesic representative.*

This fact implies that in each isotopy class of arcs of a triangulation, there is a unique geodesic representative. Starting from the topological surface $\Sigma_{g,n}$ together with an ideal triangulation, one can assume that the latter is geodesic as soon as one has chosen a hyperbolic structure on $\Sigma_{g,n}$ (i.e. a point in the Teichmüller space of $\Sigma_{g,n}$). We now present two versions of the Teichmüller space of a surface with boundary, following [FG07].

4.5.1 The Teichmüller $\mathcal{X}$-space

Ideal hyperbolic triangles are quite rigid, in the sense that every hyperbolic triangle with geodesic sides of infinite length is isometric to the triangle T with vertices $0, 1$ and ∞ in the hyperbolic upper-half plane $\mathbb{H}$. Moreover, there is a single degree of freedom assigned to the gluing of two hyperbolic triangles along an edge, called *shear parameter*, for it corresponds to shearing one triangle with respect to the other along their common edge.

Given a triangulated topological surface with punctures and a point in its Teichmüller (i.e. a class of hyperbolic metrics), one can choose a representative of this metric, and continuously deform the edges of the triangulation into geodesics for this metric. Through this procedure, every face of the triangulation becomes an ideal hyperbolic triangle, hence every edge carries a number which is the shear parameter of the two adjacent hyperbolic triangles.

Proposition 5. *The set of these $6g-6+3s$ shear parameters indexed by the edges of the triangulation parametrizes the Teichmüller $\mathcal{X}$-space of $\Sigma_{g,n}$.*

Shear parameters can be expressed as cross-ratios, and as in Subsection 4.4, the transformation under a flip corresponds to $\mathcal{X}$ mutations, hence the name of this version of the Teichmüller space.

4.5.2 The Teichmüller $\mathcal{A}$-space

There is different version of the Teichmüller called *decorated Teichmüller space.*

Definition 22. *A horocycle at a puncture in a hyperbolic surface with punctures is the limit of a sequence of hyperbolic circles, as the centre of the latter approaches the puncture. Topologically, it is a loop circling around the puncture.*

Consider a point in the Teichmüller space of the triangulated $\Sigma_{g,n}$ and one of its hyperbolic representative. Deform the edges of the triangulation to geodesics. *Decorate* this surface by choosing a horocycle at each puncture. This decoration assigns a number to each edge of the triangulation: even if each geodesic is of infinite length, now that one has a horocycle at each puncture, one can measure the length of each geodesic arc from one intersection points with a horocycle to the other. This length is finite. This set of $6g-6+3s$ coordinates is called set of $\mathcal{A}$ coordinates since the corresponds to $\mathcal{A}$ mutations in the way it changes under a flip.

A Seiberg duality

There is a very nice occurrence of quiver mutations in physics, which goes under the name of *Seiberg duality*. The story roughly goes as follows.

A.1 Perturbative and non-perturbative regimes in QFTs, renormalisation and QCD

It is commonly believed today that our Universe is governed by four fundamental forces: electromagnetism, the weak and strong nuclear forces, and gravity. At least for the first three of them, a formalism known as Quantum Field Theories (QFTs) has proven to be very useful and efficient in order to describe and predict phenomena, with an outstanding agreement between theoretical predictions (if any) and experimental results.

The mathematical structure of a d-dimensional QFT is the data of:

- a d-dimensional manifold M,
- a set a *fields* which can for example be scalar functions on M, or sections of bundles over M, connections...,
- a functional of the fields called *action*, which determines the dynamics of the theory.

Special relativity can easily be implemented in the picture by restricting to theories symmetric under the relativistic *Poincaré group*. Making the theory a *quantum* one requires a notion of integration over a space of fields, or *path integral*. The latter is very difficult to define rigorously, especially in dimension greater than 2. Besides, there may also be some constants, on which the theory depends, such as *masses* assigned to the fields, *charges*, or *coupling constants*.

However, these constants are not truly constants, but instead they depend on some *characteristic scales* of the phenomena under consideration. This very important physical phenomenon is called *renormalisation*. For example, it is known (and experimentally verified) that the mass - or the charge - of a real-world electron depends on the energy scale at which one is considering this particle. As strange and obscure this idea might seem, it is essential to our current understanding of particle physics and everything arising from it.

Some theories have a very simple structure and describe *free fields*, which are non-interacting. These can be solved exactly. However, as soon as one is not considering free fields but interacting fields (which is always the case in real life), things get more complicated, and it is not as easy as in the free theory, to predict physical phenomena.

- If the theory is close to be free, it is said to be *weakly coupled*. In that case, powerful techniques known as *perturbation theory* (in the sense that one is perturbing the nice free theory) can be used to compute physical results. This is the most successful realm of quantum field theories; the place where some of the most precise results of all of physics history have been obtained. Interactions between light and matter at reasonable energies fit into this framework, for example.
- Otherwise, the theory is said to be *strongly coupled*. In general, it is very difficult say anything in that case.

The point is that the *coupling constants* that govern how not-free a theory is, also get renormalised in general. Hence the fact that a physical process can be easily described using the perturbative machinery, or not, depends on the characteristic energy scale of this process. This feature drastically impacts QCD (quantum chromodynamics), a successful QFT that aims to describe strong interactions.

Technically speaking, QCD is an SU(3)-gauge theory with fermionic matter. Physically speaking, it describes two types of particles: *gluons*, which mediate the strong interaction (there are eight different type of gluons), and *quarks*, some kind of electron-like particles, which interact via the strong force by exchanging gluons (and they are also electrically (and weakly) charged). There are six different types of quarks dubbed up, down, charm, strange, top and bottom. Quarks have masses, and there is a *single coupling constant* which expresses how intensely quarks couple to gluons. Actually, gluons can also self-couple, but luckily it is the same coupling constant that expresses how strong this self-coupling is.

The higher the energy (or the smaller the distance), and the more QCD looks like a free theory. At high-enough energies (for example, as one smashes two atom nuclei against each other), perturbation techniques efficiently describe strong interactions: the behaviour of quarks and gluons is closed to that of free particles. One can parametrize, describe and predict the deviations from the free theory, and make theoretical predictions that, for now, are in an incredible agreement with experiments.

Conversely, under some energy threshold the theory becomes strongly coupled, and behaves in a way that is very different from a free theory. For example, around zero energy (at rest), quarks bind together to form protons, neutrons, mesons... in such a way that it becomes impossible to observe isolated quarks. This latter property is called *confinement*. Let us emphasize again how much this non-perturbative regime is very difficult to handle. There are some way to peep behind the veil, however technicalities - such as the computational power needed, makes it very difficult to study interesting real-life phenomena in a theory such as QCD.

A.2 SQCD and Seiberg duality

In order to gain insight on the phenomena appearing in the strongly-coupled regime of QCD, it is tempting to look at toy models, which are either simpler than QCD (at the cost of being less interesting), or more special (at the cost of being further away from real-world physics), in order to let one dream of easier and fruitful insights on strong-coupling effects.

Amongst all the possible toy models that are close to QCD, there is a family of theories which are more complicated than QCD, however they are way more symmetric, hence more special. These theories are generically called super-QCD, or sQCD, since they share a very interesting property called *supersymmetry*. sQCD theories are parametrised by two numbers: N and F. The first one is the *number of color*, and the second one the *number of flavors*.

The theory corresponding to (N, F) is a 4d SU(N) supersymmetric gauge theory, with $\mathcal{N} = 1$ supersymmetry. It describes $N^2 - 1$ gluon-like particles (still called gluons), and $N^2 - 1$ electron-like particles called *gluinos* (the superpartners of the gluons); as well as F quark-like particles (still called quarks), and F other particles called *squarks* (the superpartners of the quarks). That's how sQCD theories are *more* complicated that QCD: they need more particles to work. However, they are also slightly simpler than QCD, because the quarks in the former are taken to be massless.

Supersymmetry allows a better understanding of the low-energy limit of these theories, and of phenomena such as the confinement described above. Under the consensual belief that SU(N) SYM - which is $(N, 0)$ sQCD - has a *mass gap*, one can distinguished different regimes. As described and studied in [Ber19], the strategy is to fix N and let F vary:

- For $F < N$, at low energies quantum effects generate an effective runaway superpotential which makes the theory ill-defined on its own, in the sense that it is impossible to choose a consistent vacuum state for the theory. This superpotential is called *ADS superpotential* in reference to Affleck, Dine and Seiberg who first studied the phenomenon.
- For $F = N$ and $F = N + 1$, the low-energy limit of the theory is well defined, and there is even a moduli space of vacua, i.e. degrees of freedom (which appear at low-energy) in the choice of the vacuum of the theory. Some choices of vacua exhibit interesting behaviours, such as confinement.
- For $F > N + 1$ (and in fact $F \geq N + 1$), Seiberg showed that the low energy physics of the (N, F) sQCD theory can be equivalently described by the low energy physics of another theory, dubbed the *magnetic dual* theory. To some aims, using the dual description is way easier than using the original one. For example, for $\frac{3N}{2} > F > N + 1$, the (N, F)-sQCD is said to be in its *magnetic free phase*, meaning that the dual magnetic theory is - at low energy - a free theory in this regime (whereas the low energy of the original (N, F)-sQCD is strongly coupled) ! The fact that the original sQCD and the second theory describe the same physics at low energy is called *Seiberg duality.*

The magnetic dual theory of the (N, F)-sQCD is very similar to $(N - F, F)$-sQCD, but it does have an additional interaction term. Seiberg duality is an involution in the space of theories, and it can be enhanced and improved in different ways (see [Str05] for instance, and references therein).

A.3 Quiver gauge theories, mutations and string theory

Some particular supersymmetric gauge theories have the nice property that their structure can be encoded in a quiver. They are judiciously named *quiver gauge theories.* Each node in the quiver corresponds to an SU(N) gauge group - and as such, carries a natural number as label: the rank of the gauge group. Each arrow represents a bi-fundamental chiral field. Let's not spend too much time on these structures. For these theories, *Seiberg dualities correspond to mutations of the quiver.*

Summing up, the seemingly ad-hoc defined concept of mutation for quivers actually has an occurrence in physics as a relationship between QFTs, which is such that the low energy physics described by these theories is the same.

Interesting quiver gauge theories can be obtained in type IIB string theory, as the low-energy limit of some brane configurations, in a geometric background of the form $\mathbb{R}^{1,3} \times X$, where X is a Calabi-Yau 3 fold. In this framework, mutations can be given a geometric meaning in terms of brane-exchanges.

There is something that I found even more amazing. In the sQCD case, the magnetic dual of the (N, F) sQCD theory is the $(N - F, F)$ sQCD plus interaction term that we were speaking of earlier. In the case of quiver gauge theories, the additional interaction terms are already present and form a *superpotential.* The superpotential changes in a specific way under mutations, but let's not concentrate on that.

Instead, let us say that there is a number of flavors for each gauge group in the case of quiver gauge theories. Let's number $1, 2, ..., n$ the vertices of the quiver, and let $N_1, ...N_n$ be the rank of the corresponding gauge groups. The theory associated with a quiver suffers *gauge anomalies* (which makes it inconsistent) if:

$$\sum_{i \to j} N_j = \sum_{i \leftarrow j} N_j \tag{31}$$

does not hold. Hence let us assume that Eq. 31 hold. In that case, the number of flavors corresponding to the vertex i is

$$\sum_{i\to j} N_j = \sum_{i\leftarrow j} N_j \ . \tag{32}$$

Seiberg duality at the vertex i of that quiver transforms the ranks of gauge groups in the following way:

$$N'_k = \begin{cases} N_k & \text{if } k \neq i \\ \max(\sum_{i\to j} N_j, \sum_{i\leftarrow j} N_j) & \text{if } k = i \end{cases} \ , \tag{33}$$

which is written under a form that, at the risk of acting as if we had forgotten Eq. 32, emphasized the fact that this formula is nothing but the tropical version of the mutation formula of type $\mathcal{A}$, first given in Eq. 4 in the paper!

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