Characters of $\mathrm{GL}_n(\mathbb{F}_q)$ in terms of symmetric functions

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Abstract

This paper contains the notes of the third session of the seminar "Complex representations of the groups $\mathrm{GL}_n(\mathbb{F}_q)$" held at IRMA in the fall of 2018. The first session was about the ring of symmetric functions and the complex representations of the symmetric groups $\mathfrak{S}_n$ as well as the complex representations of the general linear groups $\mathrm{GL}_n(\mathbb{C})$, and the second one about Hall bialgebras of discrete valuation rings. To construct the irreducible characters of the groups $\mathrm{GL}_n(\mathbb{F}_q)$, we mainly follow [Mac98] which gives a modern presentation of the ideas first developed in [Gre55]. The paper [SZ84] follows a Hopf algebras approach to Green's results.

The cornerstone of Green's method is the definition of a map called the *characteristic map*. This idea appears in an easier framework that is one particular way of constructing the irreducible characters of the symmetric groups $\mathfrak{S}_n$, hence we briefly recall the latter before starting to work on the more complicated case of the $\mathrm{GL}_n(\mathbb{F}_q)$'s.

Contents

1 Notations and introduction

1.1 Partitions and symmetric functions

Partitions A partition $\lambda = (\lambda_i)_{i \geq 1}$ is a weakly-decreasing sequence of non-negative integers that is stationary at zero. The integer $l(\lambda) = \max\{i | \lambda_i > 0\}$ is called the length of λ, while the sum $n = \sum_i \lambda_i$ is called the weight of λ. One also says that λ is a partition of n. Sometimes it is interesting to write a partition λ as $\lambda = (1^{m_1} 2^{m_2} ... i^{m_i} ...)$ where $m_i = \mathrm{Card}\{j | \lambda_j = i\}$. Let

$\mathcal{P}$ be the set of all partitions and $\mathcal{P}_n$ the set of partitions of n.

Partitions can be schematically represented by diagrams. Formally, the diagram corresponding to a partition λ is the subset of $\mathbb{Z}^2$ of points (i,j) such that $1 \leq j \leq \lambda_i$. Conventionally, the index i increases as one goes downwards while j increases as one goes from left to right (as matrices). If one replaces each node of $\mathbb{Z}^2$ by a box, the diagram of λ is a set of r rows of boxes ordered from top to bottom and aligned on the left, such that the i-th row has λ_i boxes.

The conjugate of a partition λ is the partition denoted λ', where λ'_i is the number of boxes in the i-th column of the diagram of λ. Equivalently,

$$\lambda'_i = \mathrm{Card}\{j|\lambda_j \geq i\} \ .$$

The diagram of λ' is the transpose of the diagram of λ.

For any partition λ, set

$$n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i = \sum_{i\geq 1}\binom{\lambda'_i}{2}$$

Observe that $n(\lambda)$ is the number obtained by putting a zero in each box of the first line of the diagram of λ, a one in each box of the second line, and so on, and summing all numbers in the boxes. The first equality appears when one does a row-wise summation, and the second one when one does a column-wise summation.

Let λ be a partition. The hook-length of λ at $x = (i,j) \in \mathbb{Z}^2$ is

$$h(x) = \lambda_i + \lambda'_j - i - j + 1$$

and one has (see [Mac98], I, 1. Ex.2):

$$\sum_{x\in\lambda} h(x) = |\lambda| + n(\lambda) + n(\lambda') \ . \tag{1}$$

The ring of symmetric functions Let n be a strictly positive integer, and let

$$\Lambda_n = \mathbb{Z}[x_1, ..., x_n]^{\mathfrak{S}_n}$$

be the subring of $\mathbb{Z}[x_1, ..., x_n]$ of polynomials invariant under any permutation of the variables. Let $k \in \mathbb{N}$ and let Λ_n^k be the subspace of Λ_n consisting of homogeneous symmetric polynomials of degree k. For fixed k, and $m \geq n$, there is a surjective natural map $\Lambda_m^k \to \Lambda_n^k$ which is the identity on the x_i for $i \leq n$ and sends all other x_i's to 0. The Λ_n^k together with those maps form an inverse system, hence one can set

$$\Lambda^k = \lim_{\leftarrow} \Lambda_n^k \subset \prod_n \Lambda_n^k$$

This inverse limit projects to each of the Λ_n^k and this projection ρ_n^k is an isomorphism for all $n \geq k$. The space Λ^k has a basis consisting of the *monomial symmetric functions* m_λ, for all $\lambda \in \mathcal{P}_k$, defined by $\rho_k^k(m_\lambda)(x_1, ..., x_k) = \sum_\alpha x^\alpha$, with the sum running over all permutations α of $(\lambda_1, ..., \lambda_{l(\lambda)})$, and with $x^{(l_1,...,l_r)} := x_1^{l_1}...x_r^{l_r}$. Hence Λ^k is a free $\mathbb{Z}$-module of rank $|\mathcal{P}_k|$.

Let's set $\Lambda = \bigoplus_{k\geq 0} \Lambda^k$. It is a commutative, associative, graded ring with unit. It has canonical bases denoted (m, e, h, f, p, s) (see I. 2 & 3 in [Mac98]). We will mostly use the bases e, p and s, hence we are going to recall their definition and some of their properties.

Let $r \geq 0$. The *r-th elementary symmetric function* is the sum of all products of r distinct variables x_i, i.e $e_0 = 1$ and $e_r = m_{(1^r)}$ for $r \geq 1$. The e_r form a basis of Λ. The generating function for the e_r is

$$E(t) = \sum_{r \geq 0} e_r t^r = \prod_{i \geq 1} (1 + x_i t) \tag{2}$$

Eventually, for $\lambda \in \mathcal{P}$ one sets $e_\lambda = \prod e_{\lambda_i}$.

Let $r \geq 1$. The *r-th power sum* is $p_r = m_{(r)}$. Set $p_0 = 1$. The p_r form a basis of $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$. The generating function for the p_r is

$$P(t) = \sum_{r \geq 1} p_r t^{r-1} = \sum_{i \geq 1} \frac{d}{dt} \log \frac{1}{1 - x_i t} = \frac{E'(-t)}{E(-t)} \tag{3}$$

Eventually, for $\lambda \in \mathcal{P}$ set $p_\lambda = \prod p_{\lambda_i}$.

Set $e_r = 0$ for $r < 0$. Let $\lambda \in \mathcal{P}_n$, and define the *S-function associated with* λ to be

$$s_\lambda = \det\bigl(e_{\lambda'_i - i + j}\bigr)_{1 \leq i,j \leq l(\lambda')}$$

Then $s_\lambda \in \Lambda^{|\lambda|}$. The s_λ when λ runs over $\mathcal{P}$ form a basis of Λ.

One can endow Λ with a scalar product $\langle .,. \rangle$ for which $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ for any $\lambda, \mu \in \mathcal{P}$, i.e the s_λ form an orthonormal basis of $(\Lambda, \langle .,. \rangle)$. Moreover, $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda$ where

$$z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \ . \tag{4}$$

Hall-Littlewood symmetric functions Let $\lambda \in \mathcal{P}$ of length less than n, and set

$$P_\lambda(x_1, ..., x_n; t) = \sum_{w \in \mathfrak{S}_n / \mathfrak{S}_n^\lambda} w(x_1^{\lambda_1} ... x_n^{\lambda_n} \prod_{\lambda_i > \lambda_j} \frac{x_i - t x_j}{x_i - x_j}) \ ,$$

where in the sum $\mathfrak{S}_n^\lambda$ is the stabiliser of λ in $\mathfrak{S}_n$, and where the product runs over the pairs (i,j) such that $\lambda_i > \lambda_j$. One has $P_\lambda(x_1, ..., x_n; 0) = s_\lambda(x_1, ..., x_n)$ and $P_\lambda(x_1, ..., x_n; 1) = m_\lambda(x_1, ..., x_n)$ (see III., 2.3 & 2.4 in [Mac98]). Moreover

$$P_\lambda(x_1, ..., x_n, 0; t) = P_\lambda(x_1, ..., x_n; t)$$

hence one can define $P_\lambda(x; t)$ to be the element of $\Lambda[t]$ (the ring of symmetric functions with coefficients in $\mathbb{Z}[t]$) that projects to $P_\lambda(x_1, ..., x_n; t)$ for any $n \geq l(\lambda)$. The function $P_\lambda(x; t)$ is the *Hall-Littlewood symmetric function or HL-function* corresponding to the partition λ. Note that

$$P_{(1)^r}(x; t) = e_r(x) \ . \tag{5}$$

The HL-functions P_λ form a $\mathbb{Z}[t]$-basis of $\Lambda[t]$, and the latter can (as Λ) be endowed with a scalar product $\langle .,. \rangle$ defined in III. 4 in [Mac98].

Eventually, set $Q_\lambda(x; t) = b_\lambda(t) P_\lambda(x; t)$ where $b_\lambda(t) = \prod_{i \geq 1} \phi_{m_i(\lambda)}(t)$, and $\phi_r(t) := (1 - t)(1 - t^2)...(1 - t^r)$. The scalar product on $\Lambda[t]$ is such that $\langle P_\lambda, Q_\mu \rangle = \delta_{\mu\nu}$ (see III. 4.9 in [Mac98]).

1.2 Hall bialgebra of a discrete valuation ring

Let $\mathfrak{o}$ be a (commutative) discrete valuation ring, $\mathfrak{p}$ its maximal ideal and $k = \mathfrak{o}/\mathfrak{p}$ its residue field. A *finite* $\mathfrak{o}$*-module* M is an $\mathfrak{o}$-module M that has a finite composition series; equivalently, M is finitely generated and $\mathfrak{p}^n M = 0$ for some $n \geq 0$. Since $\mathfrak{o}$ is a discrete valuation ring, it is a PID, hence M can be written as a direct sum

$$M = \bigoplus_{i=1}^{r} \mathfrak{o}/\mathfrak{p}^{\lambda_i}$$

where the λ_i are strictly positive integers, and one can assume that they are arranged in a (weakly) decreasing order. Then $\lambda = (\lambda_1, ..., \lambda_r, 0, 0, ...)$ is a partition. One can see that $\mu_i = \dim_k(\mathfrak{p}^{i-1}M/\mathfrak{p}^i M) = \mathrm{Card}\{j|\lambda_j = i\}$ i.e $\mu = (\mu_1, ...) = \lambda'$. Hence λ is completely determined by the module M. It is called the *type* of M. If λ is a partition of n then M is said to have length n (it is the length of a composition series of M.) A submodule N of M is said to be of cotype μ in M if M/N is of type μ.

Suppose now that k is the finite field with q elements. One can show that the number $a_\lambda(q)$ of automorphisms of a finite $\mathfrak{o}$-module M of type λ is (see II. 1.6 in [Mac98])

$$a_\lambda(q) = q^{|\lambda|+2n(\lambda)} \prod_{i\geq 1} \phi_{m_i(\lambda)}(q^{-1}) ,$$

where $\phi_r(t) = (1-t)(1-t^2)...(1-t^r)$ as in the definition of the HL-function $Q_\lambda(x;t)$.

Now take M to be a finite $\mathfrak{o}$-module of type λ and let $G^\lambda_{\mu\nu}(\mathfrak{o})$ be the number of submodules of M of type ν and cotype μ. Of course, $G^\lambda_{\mu\nu}(\mathfrak{o}) = 0$ unless $|\lambda| = |\mu| + |\nu|$. Consider a free $\mathbb{Z}$-module on a basis (u_λ) indexed by the partitions, and define a product by

$$u_\lambda u_\nu = \sum_\lambda G^\lambda_{\mu\nu}(\mathfrak{o}) u_\lambda$$

The sum is finite, and this $\mathbb{Z}$-algebra is called the *Hall algebra associated with the discrete valuation ring* $\mathfrak{o}$, and is denoted $H(\mathfrak{o})$. Multiplication is commutative and associative, and has a unit element.

From III. 3.4 in [Mac98], we have the following structure theorem for $H(\mathfrak{o})$:

Proposition 1. *Let* $\psi : H(\mathfrak{o}) \otimes \mathbb{Q} \to \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ *be the* $\mathbb{Q}$*-linear mapping defined by:*

$$\psi(u_\lambda) = q^{-n(\lambda)} P_\lambda(x; q^{-1})$$

Then ψ *is an isomorphism of rings.*

The Hall algebra actually has a Hopf algebra structure defined (for example) in [Sch06].

1.3 Finite fields and orbits of the Frobenius map

Throughout the paper, k will be the unique finite field with q elements ($q = p^\alpha$ for a prime p). Fix an algebraic closure $\overline{k}$ of k, and let

$$F : x \mapsto x^q$$

be the Frobenius automorphism of $\overline{k}$ over k. For all $n \in \mathbb{N}_{>0}$, let k_n be the subfield of $\overline{k}$ consisting of the points fixed by F^n, i.e $k_n = \mathbb{F}_{q^n}$. It is the unique extension of k of degree n.

Let M be the multiplicative group of $\overline{k}$, and $M_n = M^{F^n}$ the multiplicative group of k_n. Two integers m, n such that m divides n define a norm homomorphism $N_{n,m}$:

$$\begin{array}{ccl} M_n & \to & M_m \\ x & \mapsto & x^{(q^n-1)/(q^m-1)} = \prod_{i=0}^{n/m-1} F^{mi}x \end{array} \tag{6}$$

and the groups M_n together with those maps form an inverse system. Let

$$K = \lim_{\leftarrow} M_n \ .$$

Since K is abelian and profinite, its character group $L = \hat{K}$ is discrete, and

$$L = \lim_{\rightarrow} \hat{M}_n \ .$$

The Frobenius map F acts on L. For all $n \in \mathbb{N}_{>0}$, let $L_n = L^{F^n}$ be the subgroup of L fixed by F^n, so that

$$L = \bigcup_{n \geq 1} L_n \ .$$

Moreover, $L_m \subset L_n$ if and only if m divides n. Each canonical mapping $\hat{M}_n \to L$ is an isomorphism onto L_n. Hence we get a pairing between L_n and M_n:

$$\langle \xi, x \rangle_n = \xi(x)$$

for $\xi \in L_n$ and $x \in M_n$ (this pairing is obviously not coherent with the inverse and direct systems that form (M_n) and (L_n): if m is a multiple of n then $\langle \xi, x \rangle_m = (\langle \xi, x \rangle_n)^{m/n}$).

Let Φ be the set of Frobenius orbits in M. Any such orbit is of the form $\{x, x^q, x^{q^2}, ..., x^{q^{d-1}}\}$, where $x^{q^d} = x$. The polynomial defined by

$$f(t) = \prod_{i=1}^{d-1} (t - x^{q^i})$$

is a monic irreducible polynomial in $k[t]$. Conversely, the roots of every monic irreducible polynomial $f \in k[t]$ (but t) form an F-orbit in M. We will denote with the same letter any monic irreducible polynomial (different from t), and the corresponding F-orbit of its roots in M.

Example 1. *Consider the F-orbits in M_2. By definition, M_2 is the splitting field of $t^{q^2} - t$ hence the orbits in $\Phi \cap M_2$ have either length 1 or length 2. The orbits of length 1 correspond bijectively to the non-zero elements of k, hence there are $\frac{q^2-q}{2}$ orbits of length 2, and the same number of irreducible monic polynomials of degree 2 in $k[t]$. Similarly, there are $\frac{q^3-q}{3}$ irreducible monic polynomials of degree 3 in $k[t]$. Now consider M_4. The F-orbits may have length 1, 2 or 4, but the orbits of length 1 correspond bijectively to the non-zero elements of k and we know the number of orbits of length 2, hence there are*

$$\frac{q^4 - 2 \cdot q(q-1)/2 - q}{4} = \frac{q^4}{4} - \frac{q^2}{4} + \frac{q}{4}$$

irreducible monic polynomials of degree 4 in $k[t]$. In general (see the very end of 1. in [Gre55]), there are

$$w(d, q) = \frac{1}{d} \sum_{k|d} \mu(k) q^{d/k}$$

irreducible monic polynomials of degree d in $\mathbb{F}_q$.

1.4 General strategy

One fixes the finite group $k = \mathbb{F}_q$ throughout the proof. The groups M, L as well as the set of Frobenius orbits in M constructed above of course depend on q, but since we are not going to vary the base field, we do not explicitly write this dependence to simplify the notations (they are already messy enough).

First one constructs two graded $\mathbb{C}$-algebras. On the one hand, there is the $\mathbb{C}$-algebra $A = \bigoplus A_n$ generated by the characters of the groups $\mathrm{GL}_n(k)$ for $n \in \mathbb{N}$. The multiplication in A is given by Harish-Chandra induction. As for the symmetric groups (see the first few lines of the first chapter in [Zel81] *"After classical works of A. Young it becomes clear that one must study complex representations of all these* [symmetric] *groups together, taking into account their interaction. This interaction is carried out by operations of induction and restriction."*), one has to work out the representation theory of the $\mathrm{GL}_n(k)$ by considering the family of these groups for $n \in \mathbb{N}$ with their interaction (the multiplication product on A). The $\mathbb{C}$-algebra A has a hermitian product: one requires that the graded parts are orthogonal one to another, and that on the n-th graded part it coincides with the one defined by the $\mathrm{GL}_n(k)$. Moreover, A is closely related to a tensor product of Hall algebras.

On the other hand, one constructs a $\mathbb{C}$-algebra B of symmetric functions. It is a product of disctinct copies of Λ, one for each each point in Φ (i.e, one for each irreducible monic polynomial $k[t]\backslash\{t\}$), viz.

$$B = \mathbb{C}[e_n(f) | n \in \mathbb{N}_{>0}, f \in \Phi]$$

where $e_n(f) = e_n(X_{i,f})$ with for each $f \in \Phi$, $\{X_{i,f}\}$ a countable infinite set of independent variables over $\mathbb{C}$ that have degree the degree of f as a polynomial. It induces a grading on B. The C-algebra B also has a hermitian product.

The first step of the proof is to construct a *characteristic map*

$$\mathrm{ch} : A \to B$$

that it is an isometric isomorphism of $\mathbb{C}$-algebra. Then, one uses the following theorem:

Theorem 1. *Let θ be a character of the abelian group M, G a finite group and $\rho : G \to \mathrm{GL}_n(k)$ a modular representation of G. For $x \in G$ and $(1 \geq i \geq n)$, let $u_i(x)$ be the eigenvalues of $\rho(x)$. Let $f \in \mathbb{Z}[t_1, ..., t_n]^{\mathfrak{S}_n}$ be any symmetric polynomial. Then*

$$\chi : x \mapsto f(\theta(u_1(x)), ..., \theta(u_n(x)))$$

is a character of G, i.e an integral combination of characters of complex representations of G.

This theorem holds for any θ, and since the identity map $\mathrm{GL}_n(k) \to \mathrm{GL}_n(k)$ is a modular representation of $\mathrm{GL}_n(k)$ one gets a bunch a complex characters of $\mathrm{GL}_n(k)$. In particular, one can choose and injective θ such that some symmetric functions denoted $e_k(\phi) \in B$ (for $k \in \mathbb{N}$ and ϕ a Frobenius orbit in L) are the characteristic of complex characters of $\mathrm{GL}_n(k)$'s.

The last step of the proof aims to show that that there are some linear combinations of the $e_k(\phi)$ with integer coefficients, denoted $S_{\underline{\lambda}}$ for $\underline{\lambda} : \Theta \to \mathcal{P}$ (and with Θ the set of F-orbits in L), such that

$$\langle S_{\underline{\lambda}}, S_{\underline{\kappa}} \rangle = \delta_{\underline{\lambda}\underline{\kappa}} \ ,$$

and that there are exactly the same number of $S_{\underline{\lambda}}$ as the number of conjugacy classes in $\mathrm{GL}_n(k)$. Hence one has all the irreducible characters, hence all the characters. The last thing to check is that the $\chi^{\underline{\lambda}} = \mathrm{ch}^{-1}(S_{\underline{\lambda}})$ (and not the $-\chi^{\underline{\lambda}}$) are the irreducible characters of $\mathrm{GL}_n(k)$. This is done by computing the value of $\chi^{\underline{\lambda}}$ on the conjugacy class of identity in $\mathrm{GL}_n(k)$, that is, the degree $d_{\underline{\lambda}}$ of the character $\chi^{\underline{\lambda}}$.

2 Construction of the irreducible characters of $\mathfrak{S}_n$

The ring of characters of $\mathfrak{S}_n$ Let $n \in \mathbb{N}$ and $\sigma \in \mathfrak{S}_n$. This permutation can be written as a product of disjoint cycles of respective orders $\rho_1, \rho_2, ..., \rho_k$. Those cycles commute since they have disjoint support, hence one can assume that $\rho_1 \geq \rho_2 \geq ... \geq \rho_k$, or, equivalently, that $\rho(\sigma) = (\rho_1, \rho_2, ..., \rho_k)$ is a partition of n called the type of σ.

Any two permutations of the same type are conjugate in $\mathfrak{S}_n$, and since conjugation preserves the type, the set of conjugacy classes of $\mathfrak{S}_n$ correspond bijectively to $\mathcal{P}_n$. Let $\psi : \mathfrak{S}_n \to \Lambda^n$ be defined as

$$\psi : \sigma \mapsto p_{\rho(\sigma)} \ .$$

Let $m, n \in \mathbb{N}$, and embed $\mathfrak{S}_m \times \mathfrak{S}_n$ in $\mathfrak{S}_{m+n}$ be making $\mathfrak{S}_m$ and $\mathfrak{S}_n$ act on complementary subsets of $[|1, m+n|]$. There are $(m+n)(m+n-1)/2$ different possible choices, however, the resulting subgroups in $\mathfrak{S}_{m+n}$ are all conjugate. In particular, for $v \in \mathfrak{S}_m$ and $w \in \mathfrak{S}_n$, the type of the image of the couple (v, w) in $\mathfrak{S}_{m+n}$ is well-defined, and for any choice of $v \times w$:

$$\psi(v \times w) = \psi(v) \times \psi(w) \ .$$

Let R^n be the $\mathbb{Z}$-module generated by the irreducible (complex) characters of $\mathfrak{S}_n$, and let

$$R = \bigoplus_{n \geq 0} R^n$$

with $R^0 = \mathbb{Z}$. One puts a commutative, associative, graded ring structure with identity on R by setting

$$f \cdot g = \mathrm{ind}_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}} (f \times g) \in R^{m+n}$$

for $f \in R^m$ and $g \in R^n$. Now, for $f = \sum f_n$ and $g = \sum g_n$ in R, the bracket

$$\langle f, g \rangle = \sum_{n \geq 0} \langle f_n, g_n \rangle_{\mathfrak{S}_n} = \sum_{n \geq 0} \left(\frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f_n(\sigma) g_n(\sigma^{-1}) \right)$$

defines a scalar product on R.

The characteristic map

Remark 1. *Let's just mention Corollary 1 of Section 13.1 in [Ser77] stating that for G a finite group, the following are equivalent:*

- *Every character of G is $\mathbb{Q}$-valued*
- *Every character of G is $\mathbb{Z}$-valued*
- *Every conjugacy class of G is rational, meaning that for every $g \in G$ and $k \in \mathbb{N}$ coprime with the order of g, the element g^k is conjugate to g.*

This is of course the case for the $\mathfrak{S}_n$.

Define a $\mathbb{Z}$-linear map $ch : R \to \Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ as follows. Let $f \in R^n$, and set

$$\mathrm{ch}(f) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(\sigma) \psi(\sigma)$$

One can see that $\langle ch(f), ch(g) \rangle = \langle f, g \rangle$ is an isometry, and according to Chapter 1, 7.3 of [Mac98], ch is in fact an isometric isomorphism of R onto Λ. Let $\lambda \in \mathcal{P}_n$ and set

$$\chi^\lambda = \det(\eta_{\lambda_i - i + j})_{1 \leq i,j \leq n} \in R^n$$

where η_n is the identity character of $\mathfrak{S}_n$ for $n \geq 0$ (and 0 otherwise), and where the product in the determinant is understood as the product in R (that is, after induction). The character $\mathrm{ch}(\chi^\lambda)$ turns out to be the Schur symmetric function s_λ, and since the set $\{s_\lambda\}_{\lambda \in \mathcal{P}_n}$ is an orthonormal basis of Λ^n, $\{\chi^\lambda\}_{\lambda \in \mathcal{P}_n}$ is exactly the set of irreducible characters of $\mathfrak{S}_n$. The character table of $\mathfrak{S}_n$ is the transition matrix $M(p, s)$.

3 Conjugacy classes of $\mathrm{GL}_n(k)$

Conjugacy classes in $\mathrm{GL}_n(k)$ and finite $k[t]$-modules Any $g \in \mathrm{GL}_n(k)$ acts on the vector space k^n by matrix multiplication on the left, and defines in this way a $k[t]$-module structure on k^n, given by

$$t \cdot v = gv \ ,$$

for $v \in k^n$. This module is denoted V_g. Given $g, h \in \mathrm{GL}_n(k)$, V_g and V_h are isomorphic as $k[t]$-modules if and only if g and h are in the same conjugacy class in $\mathrm{GL}_n(k)$. Hence the conjugacy classes in $\mathrm{GL}_n(k)$ correspond bijectively to the $k[t]$-modules V such that $\dim_k V = n$ and $tv = 0 \Leftrightarrow v = 0$.

Finite $k[t]$-modules and partitions Let V be an $k[t]$-module corresponding to some conjugacy class in $\mathrm{GL}_n(k)$. Since $k[t]$ is a PID, and since V is finite dimensional, V is a direct sum of cyclic modules of the form $k[t]/(f)^m$, where $m \geq 1$ and f is irreducible and monic. Hence V assigns a partition $\underline{\lambda}(f)$ to each $f \in \Phi$ such that

$$V = \bigoplus_{f,i} k[t]/(f)^{\underline{\lambda}_i(f)} \ .$$

Recall that Φ is the set of irreducible monic polynomials in $k[t]$ but t. Hence V determines a partition-valued function $\underline{\lambda}$ on Φ. Since $\dim_k k[t]/(f)^{\underline{\lambda}_i(f)} = d(f)|\underline{\lambda}_i(f)|$, with $d(f)$ the degree of f, $\underline{\lambda}$ has to satisfy

$$||\underline{\lambda}|| = \sum_{f \in \Phi} d(f)|\underline{\lambda}(f)| = n \ .$$

Conversely, any such function $\underline{\lambda} : \Phi \to \mathcal{P}$ determines a $k[t]$-module $V_{\underline{\lambda}}$ that is n-dimensional and such that $tv = 0 \Leftrightarrow v = 0$, hence a conjugacy class $c_{\underline{\lambda}}$ in $\mathrm{GL}_n(k)$. Let $c(n, q)$ be the number of conjugacy classes of $\mathrm{GL}_n(\mathbb{F}_q)$. Fairly intuitively (see the end of 1. in [Gre55]), the generating function of the $c(n, q)$ is

$$\sum_{n=0}^{\infty} c(n, q)x^n = \prod_{d=1}^{\infty} p(x^d)^{w(d,q)}$$

with $w(d, q)$ the number of irreducible monic polynomials in $k[t]$ of degree d that are not t (see Ex. 1), and

$$p(x) = \sum_{n=0}^{\infty} p_n x^n = \prod_{i=0}^{\infty} \frac{1}{1 - x^i}$$

is the generating function of the sequence of the number p_n of partitions of n.

Automorphisms of the finite $k[t]$-modules Let $f \in \Phi$, $\underline{\lambda} : \Phi \to \mathcal{P}$ and V the $k[t]$-module corresponding to $\underline{\lambda}$. Let

$$V_{(f)} = \bigoplus_{i} k[t]/(f)^{\underline{\lambda}_i(f)}$$

be the submodule of V consisting of all $v \in V$ annihilated by some power of f. The module V is the direct sum of the $V_{(f)}$ for $f \in \Phi$, and

$$\mathrm{Aut}(V) = \prod_{f \in \Phi} \mathrm{Aut}(V_{(f)}) \ .$$

The localisation $k[t]_{(f)}$ of $k[t]$ at the prime ideal (f) is a discrete valuation ring with residue field $k[t]/(f)$ which is the unique degree-$d(f)$ extension of k, that is, the finite field $\mathbb{F}_{q^{d(f)}}$.

The $k[t]$-module $V_{(f)}$ is also an $k[t]_{(f)}$-module of type $\lambda(f)$. The cardinal of the automorphism group of such a module can be computed (see Chapter 2, 1.6 in [Mac98]) and is equal to

$$|\mathrm{Aut}(V_{(f)})| = q_f^{|\underline{\lambda}(f)|+2n(\underline{\lambda}(f))} \prod_{i\geq 1} \phi_{m_i(\underline{\lambda}(f))}(q_f^{-1})$$

where $q_f = q^{d(f)} = |\mathbb{F}_{q^{d(f)}}|$. Note that the automorphisms of a $k[t]$-module V_g for some $g \in k$ are exactly the elements of $\mathrm{GL}_n(k)$ which commute with g, hence the order of the centraliser of an element whose conjugacy class is $c_{\underline{\lambda}}$ is

$$|\mathcal{C}_{\mathrm{GL}_n(k)}(c_{\underline{\lambda}})| = \prod_{f\in\Phi} |\mathrm{Aut}(V_{(f)})| := a_{\underline{\lambda}} \ . \tag{7}$$

4 The characteristic map $\mathrm{ch} : A \to B$

4.1 Parabolic induction, the ring of characters A, and Hall bialgebras

Let A_n be the space of complex-valued class functions on $\mathrm{GL}_n(k)$. It is a finite-dimensional complex vector space endowed with a hermitian product, given by

$$\langle u, v\rangle = \frac{1}{|\mathrm{GL}_n(k)|} \sum_{x\in \mathrm{GL}_n(k)} u(g)\overline{v(g)} \tag{8}$$

for $u, v \in A_n$. Let

$$A = \bigoplus_{n\geq 0} A_n$$

with the understanding that $A_0 = \mathbb{C}$. We extend the scalar product in Eq. 8 on the whole space A by requiring that the subspaces A_n be mutually orthogonal.
Let R_n be the lattice in A_n generated by the characters of $\mathrm{GL}_n(k)$, and let

$$R = \bigoplus_{n\geq 0} R_n \ .$$

The irreducible characters form an orthonormal basis of R_n, and $A_n = R_n \otimes \mathbb{C}$. -

Let $r \in [|1, n|]$ and let $n_1, n_2, ..., n_r$ be positive integers such that $\sum n_i = n$. Let P be the parabolic subgroup of $\mathrm{GL}_n(k)$ consisting of the matrices of the form

$$g = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1r} \\ 0 & g_{22} & \cdots & g_{2r} \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \ldots & g_{rr} \end{pmatrix}$$

where for all $1 \leq i \leq r$, $g_{ii} \in \mathrm{GL}_{n_i}(\mathbb{F}_q)$.

Let $u_i \in A_{n_i}$ for all $1 \leq i \leq r$. The function $u = u_1 \times u_2 \times ...u_r$ on P, defined by

$$u(g) = \prod_{i=1}^{r} u_i(g_{ii})$$

is a class function on P, hence a complex linear combination of irreducible characters (of P). In general, Frobenius formula teaches that:

$$\chi_{\mathrm{Ind}_H^G(\pi)}(g) = \sum_{x\in G/H} \chi_\pi(x^{-1}gx) \ ,$$

for $H < G$ two finite groups, π a representation of H, χ_π the character of π and $\chi_{Ind_H^G(\pi)}(g)$ the character of the representation of G induced by π. Since the RHS term is linear in χ_π, one sets:

$$u_1 \circ u_2 \circ ... \circ u_r(g) = \mathrm{Ind}_P^{\mathrm{GL}_n(k)}(u)(g) = \sum_{x \in \mathrm{GL}_n(k)/P} u(x^{-1}gx) \ .$$

The value of the class function $u_1 \circ u_2 \circ ... \circ u_r$ on a given conjugacy class in $\mathrm{GL}_n(k)$ can be explicitly written in terms of the values of the $u_1, ..., u_n$ and the structure constants of the Hall algebras $H(k[t]_{(f)})$ associated with the discrete valuation rings $k[t]_{(f)}$ for all $f \in \Phi$. In fact,

$$A \simeq \mathbb{C} \otimes \bigotimes_{f \in \Phi} H(k[t]_{(f)})$$

where the tensor products are over $\mathbb{Z}$ (see [Mac98]).

The operation $A_n \times A_m \to A_{m+n}$ defined by $(u, v) \to u \circ v$ is hence bilinear, commutative, associative and respect the grading, moreover the product of two characters is again a character hence R is a subring of A.

4.2 The ring of symmetric functions B and the characteristic map

Definition of the characteristic map For each $f \in \Phi$, let $(X_{i,f})_{i \geq 1}$ be a set of independent variables over $\mathbb{C}$. For any $u \in \Lambda$ a symmetric function, let $u(f)$ denote the symmetric function $u(X_{i,f})$. In particular, let $e_n(f) =$ be the n-th elementary symmetric functions of the $X_{i,f}$, and $p_n(f) = \sum_i X_{i,f}^n$. Let

$$B = \mathbb{C}[e_n(f) | n \geq 1, f \in \Phi] \ .$$

Let's grade B by assigning the degree $d(f)$ to each variable $X_{i,f}$, so that $e_n(f)$ is homogeneous of degree $n \cdot d(f)$. Let $\lambda \in \mathcal{P}$ and $f \in \Phi$. Set

$$\tilde{P}_\lambda(f) = q_f^{-n(\lambda)} P_\lambda(X_f, q_f^{-1})$$

$$\tilde{Q}_\lambda(f) = q_f^{|\lambda|+n(\lambda)} Q_\lambda(X_f, q_f^{-1}) = a_\lambda(q_f) \tilde{P}_\lambda(f) \ ,$$

where $q_f = q^{d(f)}$. For each $\underline{\lambda} : \Phi \to \mathcal{P}$ such that $||\underline{\lambda}|| < \infty$, let

$$\tilde{P}_{\underline{\lambda}} = \prod_{f \in \Phi} \tilde{P}_{\underline{\lambda}(f)}(f)$$

$$\tilde{Q}_{\underline{\lambda}} = \prod_{f \in \Phi} \tilde{Q}_{\underline{\lambda}(f)}(f) = a_{\underline{\lambda}} \tilde{P}_{\underline{\lambda}} \ .$$

Since $||\underline{\lambda}|| < \infty$, almost all of the terms in these products are equal to 1. The symmetric functions $\tilde{P}_{\underline{\lambda}}$ and $\tilde{Q}_{\underline{\lambda}}$ are homogeneous elements of B of degree $||\underline{\lambda}||$. According to [Mac98], III, (2.7), $(\tilde{P}_{\underline{\lambda}})$ and $(\tilde{Q}_{\underline{\lambda}})$ are $\mathbb{C}$-bases of B. One can define a hermitian product on B by setting

$$\langle \tilde{P}_{\underline{\lambda}}, \tilde{Q}_{\underline{\mu}} \rangle = \delta_{\underline{\lambda}\underline{\mu}}$$

for all $\underline{\lambda}, \underline{\mu}$.
Let $\mathrm{ch} : A \to B$ be the $\mathbb{C}$-linear mapping defined by

$$\mathrm{ch}(\pi_{\underline{\lambda}}) = \tilde{P}_{\underline{\lambda}}$$

where $\pi_{\underline{\lambda}}$ is the characteristic function of the conjugacy class $c_{\underline{\lambda}}$. The map ch is called the *characteristic map*. If $u \in A$, $\mathrm{ch}(u)$ is the *characteristic* of u.

Proposition 2. *The characteristic map* ch *is an isometric isomorphism of graded* $\mathbb{C}$*-algebras*

Proof. The ch map is clearly a linear isomorphism since it maps the basis $(\pi_{\underline{\lambda}})$ of A to the basis $\tilde{P}_{\underline{\lambda}}$ of B. It is clear that ch preserves the grading. Moreover, the isomorphism given in Prop. 1 and the definition of the conjugacy classes in $\mathrm{GL}_n(k)$ show that ch is a ring homomorphism, and Eq. 7 implies that it is an isometry. □

Definition of the orthonormal basis $(S_{\underline{\lambda}})$ By definition of the scalar product in $\Lambda[t]$ (see III in [Mac98]), the fact that $\langle P_\lambda(x;q^{-1}), Q_\lambda(x;q^{-1})\rangle = 1$ is equivalent to:

$$\sum_{\lambda\in\mathcal{P}} q^{|\lambda|} P_\lambda(x;q^{-1}) Q_\lambda(x;q^{-1}) = \prod_{i,j} \frac{1-x_iy_j}{1-qx_iy_j} \ ,$$

By replacing x_i by $X_{i,f}\otimes 1$, y_j by $1\otimes X_{j,f}$, and q by q_f, then considering the product over all $f\in\Phi$ on either side, and after that the logarithms, before expanding around 1 one gets to:

$$\log\left(\sum_{\underline{\lambda}} \tilde{P}_{\underline{\lambda}}\otimes \tilde{Q}_{\underline{\lambda}}\right) = \sum_{n\geq 1}\frac{1}{n}\sum_{f\in\Phi}(q_f^n-1)p_n(f)\otimes p_n(f) \ .$$

Let $x\in M$ and $f\in\Phi$ be the Frobenius orbit of x, or equivalently, the minimum polynomial for x over k. Set $\tilde{p}_n(x) = p_{n/d}(f)$ whenever n is a multiple of $d=d(f)$, and $\tilde{p}_n(x)=0$ if it is not. Hence for all $x\in M$, $\tilde{p}_n(x)\in B_n$ and $\tilde{p}_n(x)=0$ unless $x\in M_n$. With this notation:

$$\log\left(\sum_{\underline{\lambda}} \tilde{P}_{\underline{\lambda}}\otimes \tilde{Q}_{\underline{\lambda}}\right) = \sum_{n\geq 1}\frac{q^n-1}{n}\sum_{x\in M_n}\tilde{p}_n(x)\otimes\tilde{p}_n(x) \ . \tag{9}$$

Now, for all $\xi\in L$, set

$$\tilde{p}_n(\xi) = \begin{cases} (-1)^{n-1}\displaystyle\sum_{x\in M_n}\langle\xi,x\rangle_n\tilde{p}_n(x) & \text{when } \xi\in L_n \\ 0 & \text{otherwise}\end{cases} \ . \tag{10}$$

Note that since $\tilde{p}_n(x)=\tilde{p}_n(y)$ if x and y are in the same $f\in\Phi$, hence $\tilde{p}_n(\xi)=\tilde{p}_n(\eta)$ if ξ and η are in the same F-orbit in L.
Let Θ be the set of F-orbit in L, and for each $\phi\in\Theta$, let $d(\phi)$ the number of elements of ϕ. For any $\xi\in\phi$ and $r\geq 1$, set

$$p_r(\phi) = \tilde{p}_{rd(\phi)}(\xi)$$

The $p_r(\phi)$'s can be thought as the power sums of a set of variables $Y_{i,\phi}$, where each of the $Y_{i,\phi}$ has degree $d(\phi)$. All the other standard symmetric functions of the $Y_{i,\phi}$ can be defined straightforwardly in the same way, and in particular the S-functions $s_\lambda(\phi)$ (where λ runs over $\mathcal{P}$).

For any function $\underline{\lambda}:\Theta\to\mathcal{P}$ such that

$$||\underline{\lambda}|| = \sum_{\phi\in\Theta} d(\phi)|\lambda(\phi)| < \infty \ ,$$

let

$$S_{\underline{\lambda}} = \prod_{\phi\in\Theta} s_{\underline{\lambda}(\phi)}(\phi) \ .$$

It is a homogeneous symmetric function in B of degree $||\lambda||$. Now, let S be the $\mathbb{Z}$-submodule of B generated by the $S_{\underline{\lambda}}$. Schur functions have nice properties implying that S is a subring of B, and that

$$B = S\otimes_{\mathbb{Z}}\mathbb{C}$$

Proposition 3. *The $S_{\underline{\lambda}}$ form an orthonormal basis of S.*

Proof. Since

$$\log\left(\sum_\lambda s_\lambda(x)s_\lambda(y)\right) = \sum_{i,j}\log(1-x_iy_j)^{-1} = \sum_{n\geq 1}\frac{1}{n}p_n(x)p_n(y) \ ,$$

$$\log\left(\sum_{\underline{\lambda}} S_{\underline{\lambda}} \otimes S_{\underline{\lambda}}\right) = \sum_{n\geq 1}\frac{1}{n}\sum_{\phi\in\Theta} p_n(\phi)\otimes\overline{p_n(\phi)} = \sum_{n\geq 1}\frac{1}{n}\sum_{\xi\in L_n}\tilde{p}_n(\xi)\otimes\overline{\tilde{p}_n(\xi)}$$

holds, where for $u = \sum_\lambda u_{\underline{\lambda}}\tilde{P}_{\underline{\lambda}} \in B$, $\overline{u}$ denotes $\sum_\lambda \overline{u_{\underline{\lambda}}}\tilde{P}_{\underline{\lambda}}$. Subsequently:

$$\begin{aligned}\log\left(\sum_{\underline{\lambda}} S_{\underline{\lambda}} \otimes S_{\underline{\lambda}}\right) &= \sum_{n\geq 1}\frac{1}{n}\sum_{\xi\in L_n}\sum_{x,y\in M_n}\langle\xi,x\rangle_n\overline{\langle\xi,y\rangle_n}\tilde{p}_n(x)\otimes\tilde{p}_n(y)\\ &= \sum_{n\geq 1}\frac{q^n-1}{n}\sum_{x\in M_n}\tilde{p}_n(x)\otimes\tilde{p}_n(x)\end{aligned}$$

where the last equality comes from the orthogonality of the characters of the finite abelian group M_n. Comparing this last result with Eq. 9:

$$\sum_{\underline{\lambda}} S_{\underline{\lambda}} \otimes S_{\underline{\lambda}} = \sum_{\underline{\mu}}\tilde{P}_{\underline{\mu}}\otimes\tilde{Q}_{\underline{\mu}} \ . \tag{11}$$

Expressing the $S_{\underline{\lambda}}$ in the bases $(\tilde{P}_{\underline{\lambda}})$ and $(\tilde{Q}_{\underline{\lambda}})$, one gets

$$\langle S_{\underline{\lambda}}, S_{\underline{\kappa}}\rangle = \delta_{\underline{\lambda},\underline{\kappa}}$$

□

5 Characters and irreducible characters

5.1 Construction of the characters

In this section, we use the theorem 1 stated in 1.4, first proved in [Gre55] following Brauer's characterisation of characters, and rephrased in [Mac98], IV, (5.1). We use the notations of the latter reference.

Let θ be any (for now) injective character of M (an isomorphism from M onto the group of roots of unity in $\mathbb{C}$ of order prime to $p = \mathrm{char}(k)$). For each $g \in \mathrm{GL}_n(k)$, let $(x_i)_{i\in[|1;n|]}$ be the eigenvalues of g in the defining modular representation. Then from Th. 1, one learns that:

$$\chi_r : g \mapsto e_r(\theta(x_1), ..., \theta(x_n))$$

is a complex character of $\mathrm{GL}_n(k)$.

For any $f = \prod_i (t - y_i) \in \Phi$, let $\tilde{f} = \prod_i (1 + y_i t)$, and $\theta(\tilde{f}) = \prod_i (1 + \theta(y_i)t)$. Eq. 2 implies:

$$\sum_{r=0}^{n}\chi_r(g)t^r = \prod_{i=1}^{n}(1+\theta(x_i)t) \ ,$$

$$\sum_{r=0}^{n}\chi_r(g)t^r = \prod_{f\in\Phi}\theta(\tilde{f})^{|\underline{\lambda}(f)|} \ .$$

To see this, observe that the characteristic polynomial of any $g \in \mathrm{GL}_n(k)$ in the class $c_{\underline{\lambda}}$ can be expressed as (see IV, 2. Ex. 2 in [Mac98])

$$\prod_{f\in\Phi} f^{|\underline{\lambda}(f)|} = \prod_{i=1}^{n}(t - x_i) \ .$$

Write the class function χ_r in the basis of characteristic functions of conjugacy classes $(\pi_{\underline{\lambda}})$:

$$\chi_r = \sum_{\underline{\lambda}} \chi_r(c_{\underline{\lambda}}) \pi_{\underline{\lambda}}$$

with $\chi_r(c_{\underline{\lambda}})$ the value of χ_r at the class $c_{\underline{\lambda}}$. By definition of the characteristic map, $\mathrm{ch}(\pi_{\underline{\lambda}}) = \tilde{P}_{\underline{\lambda}}$, hence

$$\sum_{r=0}^{n} \mathrm{ch}(\chi_r) t^r = \sum_{\underline{\lambda}} \prod_{f \in \Phi} \theta(\tilde{f})^{|\underline{\lambda}(f)|} \tilde{P}_{\underline{\lambda}}$$

where the sum on the right runs over all the maps $\underline{\lambda} : \Phi \to \mathcal{P}$ satisfying $||\underline{\lambda}|| = n$.
Since (see III, 3, Ex. 1 of [Mac98])

$$\sum_{|\lambda|=m} t^{n(\lambda)} P_\lambda(x;t) = h_m(x)$$

one has, for each $f \in \Phi$:

$$\sum_{|\lambda|=m} \tilde{P}_\lambda(f) = h_m(f) \ .$$

Hence

$$\sum_{r=0}^{n} \mathrm{ch}(\chi_r) t^r = \sum_{\underline{\alpha}} \prod_{f \in \Phi} \theta(\tilde{f})^{\underline{\alpha}(f)} h_{\underline{\alpha}(f)}(f)$$

where the sum on the RHS runs over all the maps $\underline{\alpha} : \Phi \to \mathbb{N}$ satisfying $\sum_{f \in \Phi} d(f)\underline{\alpha}(f) = n$ hence

Proposition 4. *Let $0 \leq r \leq n$. The characteristic $\mathrm{ch}(\chi_r)$ of the character χ_r is equal to the coefficient of $t^r u^n$ in*

$$H(t,u) = \prod_{f \in \Phi} \prod_{i \geq 1} (1 - \theta(\tilde{f}) X_{i,f} u^{d(f)})^{-1}$$

A more hands-on form for $H(t,u)$ Let $m \in \mathbb{N}_{\geq 1}$. The unique degree-$m$ field extension of k is by definition the decomposition field of the polynomial $F^m x - x$, moreover

$$\{F^i\}_{0 \leq i \leq m-1} = \mathfrak{g}_m = \mathrm{Gal}(\mathbb{F}_{q^m}/k) \ .$$

This group is abelian and acts by multiplication on the set of subsets of $\mathfrak{g}_m$. For $I \subset \mathfrak{g}_m$ let $\theta_{m,I}$ be the character on M_m (identified with its image in L_m) defined by

$$\theta_{m,I} : x \mapsto \prod_{\gamma \in I} \theta(\gamma x)$$

Definition 1. *A subset $I \subset \mathfrak{g}_m$ is said to be primitive if $\mathrm{Stab}_{\mathfrak{g}_m}(\{I\}) = \{\mathrm{id}\}$, and in that case the pair (m, I) is said to be an index.*

Theorem 2. *One has*

$$H(t,u) = \prod_{[s,J]} \Big(\sum_{m \geq 0} e_m(\phi_{s,J})(t^{|J|}u^s)^m\Big) \ ,$$

where the first product runs over the set of equivalent classes of indices (s, J) (with $(s,J) \sim (s,J')$ if $\exists \gamma \in \mathfrak{g}_s$ such that $J' = \gamma J$). For each pair of integers r, n such that $0 \leq r \leq n$, the coefficient of $t^r u^n$ in is the characteristic of a character of $\mathrm{GL}_n(k)$.

Proof. Let's compute the logarithm of $H(t,u)$:

$$\log H(t,u) = \sum_{f \in \Phi} \sum_{i \geq 1} \log\Big(1 - \theta(\tilde{f}) X_{i,f} u^{d(f)}\Big)^{-1} = \sum_{m \geq 1} \frac{1}{m} \sum_{f \in \Phi} \theta(\tilde{f})^m p_m(f) u^{md(f)} \ .$$

Note that for x any root of f,

$$\theta(\tilde{f}) = \prod_{i=0}^{d(f)-1} (1 + t\theta(F^i x)) \ .$$

Since $F^{d(f)}x = x$, one has

$$\theta(\tilde{f})^m = \prod_{i=0}^{md(f)-1} (1 + t\theta(F^i x)) \ ,$$

and subsequently

$$\log H(t,u) = \sum_{m\geq 1} \frac{u^m}{m} \sum_{x\in M_m} \tilde{p}_m(x) \prod_{i=0}^{m-1} (1 + t\theta(F^i x)) \ .$$

Now, note that by Eq. 10:

$$\sum_{x\in M_m} \tilde{p}_m(x) \prod_{i=0}^{m-1} (1 + t\theta(F^i x)) = \sum_{I\subset \mathfrak{g}_m} t^{|I|} \sum_{x\in M_m} \tilde{p}_m(x) \prod_{\gamma\in I} \theta(\gamma x) = (-1)^{m-1} \sum_{I\subset \mathfrak{g}_m} t^{|I|} \tilde{p}_m(\theta_{m,I}) \ ,$$

hence

$$\log H(t,u) = \sum_{m\geq 1} \frac{(-1)^{m-1} u^m}{m} \sum_{I\subset \mathfrak{g}_m} t^{|I|} \tilde{p}_m(\theta_{m,I}) \ .$$

- Suppose that the pair (m, I) is an index. Then by definition all the *translates* γI (for $\gamma \in \mathfrak{g}_m$) are distinct, and the F-orbit of $\theta_{m,I}$ consists of m elements.
- Conversely, if I is not primitive, then $\mathrm{Stab}_{\mathfrak{g}_m}(\{I\}) = \mathfrak{h}$ for some non-trivial subgroup $\mathfrak{h}$ of $\mathfrak{g}_m$, generated by F^s for some s dividing m. Since I is stable under multiplication by elements of $\mathfrak{h}$, I is the union of cosets of $\mathfrak{h}$ in $\mathfrak{g}_m$, say $I = \mathfrak{h}J$ for $J = I/\mathfrak{h}$ a subset of $\mathfrak{g}_m/\mathfrak{h} = \mathfrak{g}_s = \mathrm{Gal}(\mathbb{F}_{q^s}/k)$. Moreover, J is by construction a primitive subset of $\mathfrak{g}_s$, or equivalently, (s, J) is an index. Thus the F-orbit $\phi_{s,J}$ of $\theta_{s,J}$ has s elements. Since $\theta_{m,I} = \theta_{s,J} \circ N_{m,s}$, where $N_{m,s}$ is the norm homomorphism defined in Eq. 6, $\theta_{m,I}$ and $\theta_{s,J}$ define the same element of L, and $\tilde{p}_m(\theta_{m,I}) = p_{m/s}(\phi_{s,J})$.

Hence $\log H(t,u)$ becomes

$$\log H(t,u) = \sum_{[s,J]} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} p_n(\phi_{s,J})(t^{|J|}u^s)^n \ ,$$

where the left-most sum runs over all equivalence classes of indices (s, J), where (s, J) and (s, J') are in the same equivalence class if J' is a translate of J, i.e if $\phi_{s,J} = \phi_{s,J'}$. Now one uses Eq. 3 to conclude the proof. □

Corollary 1. *One can choose θ so that for ϕ any F-orbit in L and any $n \geq 0$, $e_n(\phi)$ is the characteristic of a character of $G_{nd(\phi)}$.*

Proof. The proof runs by induction on the pair $(n, d(\phi))$ with lexicographic ordering.

- Let $n = d = 1$, then $\phi = \{\xi\}$ where $\xi = L_1$ is a character of M_1, and

$$e_1(\xi) = p_1(\xi) = \sum_{x\in G_1} \xi(x) p_1(x) = \mathrm{ch}(\xi)$$

  since $\mathrm{ch}(\pi_x) = p_1(x)$ for π_x the characteristic function of $\{x\}$.

- Now let $\chi_{n,d}$ be the coefficient of $t^n u^{nd}$ in $H(t,u)$. By Th. 2, $\chi_{n,d}$ is the characteristic of a character of G_{nd}, and

$$\chi_{n,d} = \sum_v \prod_{[s,J]} e_{v(s,J)}(\phi_{s,J}) \qquad (12)$$

where the first sum runs over the functions v on the set of equivalence classes of indices such that

$$\textstyle\sum_{[s,J]} |J|v(s,J) = n \ , \quad \sum_{[s,J]} sv(s,J) = nd$$

Each index (s,J) that occurs in Eq. 12 has $v(s,J) \le n$, and if equality holds then $s \le d$. Thus $(v(s,J), s) \le (n,d)$ with equality if and only if $v(s,J) = n$ (so that $|J| = 1$) and $s = d$. Hence $\chi_{n,d}$ has leading term $e_n(\phi_{d,\{\mathrm{id}\}})$ where $\phi_{d,\{\mathrm{id}\}}$ is the F-orbit of $\theta_{|M_d}$. Now one can choose θ such that $\theta_{|M_d} = \phi$, and then $\chi_{n,d} = e_n(\phi) + \ldots$, where the dots are products of at least two e's that are characteristics of characters of groups $\mathrm{GL}_{n'd'}(k)$ by inductive hypothesis, with $(n', d') < (n,d)$.

□

5.2 Irreducible characters

Theorem 3. *Let $\underline{\lambda} : \Theta \to \mathcal{P}$ such that $||\underline{\lambda}|| = n$. The corresponding $S_{\underline{\lambda}} \in B$ is the characteristic of an irreducible character of $\mathrm{GL}_n(k)$ denoted $\chi^{\underline{\lambda}}$. Moreover, every irreducible character of $\mathrm{GL}_n(k)$ is of this form.*

Proof. Each S-function s_λ is a polynomial in the e's with integer coefficients, hence by Cor. 1 each $s_\lambda(\phi)$ hence each $S_{\underline{\lambda}}$ is the characteristic of a character of $\mathrm{GL}_n(k)$. For each $\underline{\lambda} : \Theta \to \mathcal{P}$ such that $||\underline{\lambda}|| = n$, let $\chi^{\underline{\lambda}}$ be the character corresponding to $S_{\underline{\lambda}}$. Since the characteristic map is an isometric isomorphism, and since the $S_{\underline{\lambda}}$ form an orthonormal basis of the subring S of B, one has

$$\langle \chi^{\underline{\lambda}}, \chi^{\underline{\kappa}} \rangle = \delta_{\underline{\lambda},\underline{\kappa}}$$

hence each $\chi^{\underline{\lambda}}$ is (up to the sign) an irreducible character of $\mathrm{GL}_n(k)$. The question of the sign is ruled in Prop. 5. Note that since a finite group has as many irreducible characters as conjugacy classes, $\mathrm{GL}_n(k)$ has

$$\#\{\underline{\lambda} : \Phi \to \mathcal{P} |\ ||\underline{\lambda}|| = n\}$$

distinct irreducible characters. Since L and M are isomorphic (they are abelian and mutual duals), this number equals

$$\#\{\underline{\lambda} : \Theta \to \mathcal{P} |\ ||\underline{\lambda}|| = n\}$$

and the characters $\pm\chi^{\underline{\lambda}}$ exhaust all the irreducible characters of $\mathrm{GL}_n(k)$. □

Proposition 5. *Let $\underline{\lambda} : \Theta \to \mathcal{P}$ such that $||\underline{\lambda}|| = n$. Then $\chi^{\underline{\lambda}}$ is an irreducible character of $\mathrm{GL}_n(k)$. Moreover, the character table of $\mathrm{GL}_n(k)$ is the transition matrix between the bases of B consisting of the Schur functions $(S_{\underline{\lambda}})$ one the one hand, and the Hall-Littlewood polynomials $(\tilde{P}_{\underline{\lambda}})$ on the other.*

Proof. Let $\chi^{\underline{\lambda}}_{\underline{\mu}}$ be the value of $\chi^{\underline{\lambda}}$ on the class $c_{\underline{\mu}}$. Then

$$\chi^{\underline{\lambda}} = \sum_{\underline{\mu}} \chi^{\underline{\lambda}}_{\underline{\mu}} \pi_{\underline{\mu}} \ ,$$

where the sum runs over all $\underline{\mu} : \Phi \to \mathcal{P}$ such that $||\underline{\mu}|| = ||\underline{\lambda}||$. Taking the image by the characteristic map gives

$$S_{\underline{\lambda}} = \sum_{\underline{\mu}} \chi^{\underline{\lambda}}_{\underline{\mu}} \tilde{P}_{\underline{\mu}}$$

and it shows that $\chi^{\underline{\lambda}}_{\underline{\mu}} = \langle S_{\underline{\lambda}}, \tilde{Q}_{\underline{\mu}} \rangle$, hence that

$$\tilde{Q}_{\underline{\mu}} = \sum_{\underline{\lambda}} \chi^{\underline{\lambda}}_{\underline{\mu}} \overline{S_{\underline{\lambda}}} \ . \tag{13}$$

To complete the proof we have to show that $d_{\underline{\lambda}} = \chi^{\underline{\lambda}}(\mathrm{id}_{\mathrm{GL}_n(k)})$ is positive. Now consider the conjugacy class of identity $c_{\underline{\mu}}$ in $\mathrm{GL}_n(k)$, and let $f_1 = t - 1 \in k[t]$. Then $\underline{\mu}(f_1) = (1^n)$ and $\underline{\mu}(f) = 0$ for $f \neq f_1$, and from Eq. 5 it follows that

$$\tilde{Q}_{\underline{\mu}} = q^{n(n+1)/2} \phi_n(q^{-1}) e_n(f_1) = \psi_n(q) e_n(f_1) \ ,$$

with $\psi_n(q) = \prod_{i=1}^n (q^i - 1)$, and from Eq. 13 one has

$$\psi_n(q) e_n(f_1) = \sum_{||\underline{\lambda}||=n} d_{\underline{\lambda}} \overline{S_{\underline{\lambda}}} \ . \tag{14}$$

Let $\delta : B \to \mathbb{C}$ be the $\mathbb{C}$-algebra homomorphism

$$\delta(\tilde{p}_n(x)) = \begin{cases} 0 \text{ if } x \neq 1 \\ (-1)^{n-1}/(q^n - 1) \text{ if } x = 1 \end{cases}$$

which implies that $\delta(\tilde{p}_n(\xi)) = (q^n - 1)^{-1}$ for $\xi \in L_n$, hence that for any $\phi \in \Theta$,

$$\delta(p_n(\phi)) = (q^n_\phi - 1) = \sum_{i \geq 1} q_\phi^{-in}$$

where $q_\phi = q^{d(\phi)}$. Hence

$$\delta(s_\lambda(\phi)) = s_\lambda(q_\phi^{-1}, q_\phi^{-2}, \ldots) = q_\phi^{-|\lambda| - n(\lambda)} \prod_{x \in \lambda} (1 - q_\phi^{-h(x)})^{-1}$$

where $h(x)$ is the hook-length at $x \in \lambda$. From Eq. 1 it follows that

$$\delta(s_\lambda(\phi)) = q_\phi^{n(\lambda')} \prod_{x \in \lambda} (q_\phi^{h(x)} - 1)^{-1} = q_\phi^{n(\lambda')} \tilde{H}_\lambda(q_\phi)^{-1} \ . \tag{15}$$

where

$$\tilde{H}_\lambda(t)^{-1} = \prod_{x \in \lambda} (t^{h(x)} - 1)$$

Now from Eq. 9 and Eq. 11 one has

$$\log\left(\sum_{\underline{\lambda}} \delta(S_{\underline{\lambda}}) \overline{S_{\underline{\lambda}}} \right) = \log\left(\sum_{\underline{\lambda}} \delta(\tilde{P}_{\underline{\mu}}) \tilde{Q}_{\underline{\mu}} \right) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \tilde{p}_n(1) = \log \prod_i (1 + X_{i,f_1}) \ ,$$

hence

$$\sum_{\underline{\lambda}} \delta(S_{\underline{\lambda}}) \overline{S_{\underline{\lambda}}} = \sum_{n \geq 0} e_n(f_1) \text{ and } \sum_{||\underline{\lambda}||=n} \delta(S_{\underline{\lambda}}) \overline{S_{\underline{\lambda}}} = e_n(f_1) \ ,$$

and comparing with Eq. 14 one gets

$$d_{\underline{\lambda}} = \psi_n(q) \delta(S_{\underline{\lambda}}) \ .$$

This last result together with Eq. 15 eventually gives

$$d_{\underline{\lambda}} = \psi_n(q) \prod_{\phi \in \Theta} q_\phi^{n(\underline{\lambda}(\phi)')} \tilde{H}_{\underline{\lambda}(\phi)}(q_\phi)^{-1} \ ,$$

which is clearly positive. □

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